Theory of noncooperative games

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Chapter 0

Game theory and its subject (some examples)

0.1 Matrix games — Scissors-Paper-Stone

Each of two players simultaneously makes a gesture indicating one of the three objects in the name of the game (a closed fist for "stone" etc.). If they choose the same object, the game is a draw. Otherwise, the winner is decided by the rule: "Scissors cut paper, paper covers stone, stone breaks scissors". The payoff is $+1$ for a win and -1 for a loss.

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

This is a two–person zero–sum game.

0.2 Games on the unit square—classical duel

Suppose each of the players 1 and 2 has one bullet. If one player fires his bullet and misses, the other player knows at once that his opponent has no bullet left and can walk on until they are face to face and thus get a sure hit.

A strategy for player 1 is to fire when the two players are at a distance x unit apart, $0 \leq x \leq 1$. Similarly, a strategy for player 2 is to fire when they are at a distance y unit apart, $0 \leq y \leq 1$. Let the accuracy function of player 1 be $P_1(x)$, which represents the probability of player 1's hitting his opponent if he fires when they are at a distance x unit apart. Normally the accuracy function is a decreasing function of the distance. Similarly, let $P_2(y)$ be the accuracy function of player 2. Both players wish to choose an appropriate opportunity to fire their bullets. If a player fires too early, the shot may not be accurate enough. On the other hand, if he holds his fire too long his opponent may fire and hit him. The timing of the actions is decisive for both players. The expected payoff to player 1 is

$$P(x,y) = \begin{cases} 1 \cdot P_1(x) + (-1)(1-P_1(x)) & , \quad x > y \\ 1 \cdot P_1(x)[1-P_2(x)] + (-1)[1-P_1(x)] \cdot P_2(x) & , \quad x = y \\ (-1)P_2(y) + 1 \cdot (1-P_2(y)) & , \quad x < y \end{cases}$$

$$P(x,y) = \begin{cases} 2P_1(x) - 1 & , \quad x > y \\ P_1(x) - P_2(x) & , \quad x = y \\ 1 - 2P_2(y) & , \quad x < y \end{cases}$$

Under the same circumstances but if both guns are equipped with a silencer we get

$$P(x,y) = \begin{cases} 1 \cdot P_1(x) + (-1)(1-P_1(x)) \cdot P_2(y) & , \quad x > y \\ 1 \cdot P_1(x)[1-P_2(x)] + (-1)[1-P_1(x)] \cdot P_2(x) & , \quad x = y \\ (-1)P_2(y) + 1 \cdot (1-P_2(y))P_1(x) & , \quad x < y \end{cases}$$

$$P(x,y) = \begin{cases} P_1(x) - P_2(y) + P_1(x) \cdot P_2(y) & , \quad x > y \\ P_1(x) - P_2(x) & , \quad x = y \\ P_1(x) - P_2(y) - P_1(x) \cdot P_2(y) & , \quad x < y \end{cases}$$

0.3 Two–person non–zero–sum games —Prisoners dilemma

Two criminals, call them Bonnie and Clyde, are arrested for robbery. The police immediately separate them so that they are unable to communicate in any way. Each of them is offered the following deal:

If you confess and implicate the other prisoner, then you will serve only one year in jail (if the other guy doesn't confess) or five years (if the other guy does confess). On the other hand, if you dont confess and the other prisoner does, you will serve ten years. Finally, if neither of you confess, you will both serve two years since your case won't be very strong.

Viewing this as a game, we see that each player has only two strategies. Let's call them C (for "cooperate with the other criminal") and D (for "defect"). Adopting strategy C means to refuse to confess, that is, to act as though the two prisoners formed a team of people who could trust each other, while adopting D means to confess.

	C	D
C	-2	-10
D	-1	-5

⟵ Clyde's strategy

Bonnie's payoff

↑

Bonnie's strategy

↓

	C	D
C	-2	-1
D	-10	-5

⟵ Clyde's strategy

Clyde's payoff

We get a so–called bi–matrix game.

0.4 N-person cooperative games–Lake Wobegon game

The municipal government of Lake Wobegon, Minnesota, is run by a City Council and a Mayor. The Council consists of six Aldermen and a Chairman. A bill can become a law in Lake Wobegon in two ways. These are:

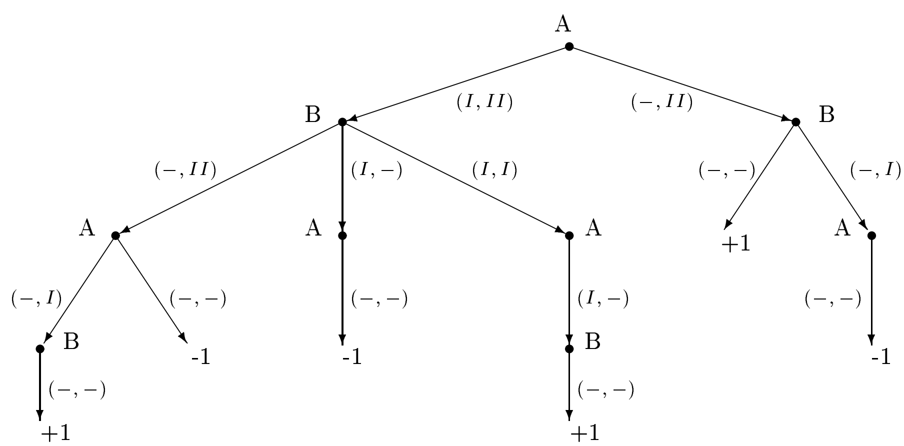
- A majority of the Council (with the Chairman voting only in case of a tie among the Aldermen) approves it and the Mayor signs it.
- The Council passes it, the Mayor vetoes it, but at least six of the seven members of the Council then vote to override the veto. (In this situation the chairman always votes.)

The game consists of all eight people involved in signifying approval or disapproval of the given bill.

In this game the players are permitted to cooperate fully. They could coordinate their strategies (make binding agreements about joint strategies). They could also agree to pool their individual payoffs and then redistribute the total in a specified way.

0.5 Games in extensive form–a nim play

There are two players, and, at the start, two piles on the table in front of them, each containing n matches. In turn, the players take any (positive) number of matches from one of the piles. The player taking the last match loses. The following figure shows a game tree in case $n = 2$.



Player B has a sure win. If A moves left from the root, B moves middle, if A moves right, B moves right.

Chapter 1

Matrix games

1.1 Finite two–person zero–sum games in normal form

Given a game $\Gamma = \langle S_1, S_2, H_1, H_2 \rangle$, where

$$\begin{array}{ll} S_1 = \{\sigma_1^1, \sigma_1^2, \ldots, \sigma_1^m\} & \text{– is the set of strategies of } P_1, \\ S_2 = \{\sigma_2^1, \sigma_2^2, \ldots, \sigma_2^n\} & \text{– is the set of strategies of } P_2. \end{array}$$

$H_1, H_2 : S_1 \times S_2 \to \mathbb{R}^1$ are the payoff–functions with

$$H_1(\sigma_1^i, \sigma_2^j) = -H_2(\sigma_1^i, \sigma_2^j), \quad \forall (\sigma_1^i, \sigma_2^j) \in S_1 \times S_2.$$

What player P_1 wins is just what player P_2 loses, and vice versa.

The games rules are the following:

- P_1 takes its choice independently of P_2 by selecting one of the elements σ_1^i from S_1.
- P_2 picks out one of the elements σ_2^j from S_2, independently of player P_1's decision.
- P_1 and P_2 have full information about the game, i.e. they know S_1, S_2, H_1 and H_2.
- Both players are interested in maximizing their own payoff $H_i(\sigma_1^i, \sigma_2^j)$.
- Both players act cautious, i.e. they will not take a risk (conservative behaviour).

We will use following notions:

S_i	– strategy set of player P_i
σ_1^i resp. σ_2^j	– pure strategy of P_1 resp. P_2
H_i	– payoff–function of P_i
(σ_1^i, σ_2^j)	– situation in game Γ

Obviously, we can abstract themselves from the real background (nature of S_1, S_2, H_1, H_2) and consider only the case $S_1 = \{1, 2, \ldots, m\}$, $S_2 = \{1, 2, \ldots, n\}$, $H(i, j) = a_{ij}$, $\forall (i, j) \in S_1 \times S_2$, where H stands for H_1.

Definition 1.1. *We say, that there is given a* **matrix game** Γ *if a real matrix* $A = \{a_{ij}\}_{i=1,j=1}^{m,n}$ *is submitted and two players* P_1 *and* P_2 *have agreed, that*
– P_1 *chooses a row* (i)
– P_2 *chooses a column* (j)
– P_1 *gains the payoff* a_{ij} *from* P_2.

What is to say about optimal behaviour? The interests of the two players are completely conflicting. If player P_1 chooses his strategy i, he can be sure to obtain at least the payoff

$$\min\{a_{ij} \;:\; 1 \le j \le n\}, \qquad (\circ)$$

which is the minimum of the i-th–row elements of the payoff matrix. Since P_1 wishes to maximize his payoff, he can choose i so as to make $(\circ)$ as large as possible. That is to say, player P_1 can choose i so as to receive a payoff not less than

$$\underline{v} := \max_{1\le i\le m} \min_{1\le j\le n} a_{ij}.$$

Similarly, if player P_2 chooses his strategy j, he will lose at most

$$\max\{a_{ij} \;:\; 1 \le i \le m\}, \qquad (\circ\circ)$$

which is the maximum of the j-th column elements in the payoff matrix. Since player P_2 wishes to minimize his payoff, he will try to choose j so as to make $(\circ\circ)$ a minimum. That is to say, P_2 can choose j so as to make his loss not greater than

$$\overline{v} := \min_{1\le j\le n} \max_{1\le i\le m} a_{ij}.$$

Lemma 1.1. *For any matrix* $A = \{a_{ij}\}_{i=1,j=1}^{m,n}$ *we have*

$$\underline{v} \le \overline{v}.$$

Examples

1. In the scissors-paper-stone game from section 0.1 we get $\underline{v} = -1$, $\overline{v} = +1$.
2. Example 2: P_1 chooses a number p from the set $\{0, 1, 2, 3\}$ and P_2 chooses a number q from the set $\{0, 1, 2\}$. P_1 gains the amount $p(q-p) + q(q+p)$ from P_2.

$p\backslash q$	0	1	2
0	0	1	4
1	-1	2	7
2	-4	1	8
3	-9	-2	7

We get

$$\underline{v} = \overline{v} = 0.$$

1.2 Solvability in pure strategies

If the equation $\underline{v} = \overline{v}$ holds, there exist an i^* and a j^* such that

$$\underline{v} = \max_i \min_j \ a_{ij} = \min_j \ a_{i^*j}$$

and

$$\overline{v} = \min_j \max_i \ a_{ij} = \max_i \ a_{ij^*} \ .$$

Hence

$$\min_j \ a_{i^*j} = \max_i \ a_{ij^*} \ . \qquad (+)$$

But

$$\min_j \ a_{i^*j} \leq a_{i^*j^*} \leq \max_i \ a_{ij^*} \ .$$

Thus, with $v := \underline{v} = \overline{v}$ we get from (+)

$$\min_j \ a_{i^*j} = a_{i^*j^*} = v = \max_i \ a_{ij^*}$$

and for all i and all j we have

$$a_{ij^*} \leq a_{i^*j^*} = v \leq a_{i^*j} \quad .$$

That is to say, if player P_1 chooses the strategy i^*, then the payoff cannot be less than v (if player P_2 departs from strategy j^*), if player P_2 chooses the strategy j^*, then the payoff cannot exceed v (if player P_1 departs from the strategy i^*).

Definition 1.2. *We call i^* and j^** **optimal strategies** *of player P_1 and player P_2 respectively and (i^*, j^*) a* **saddle point** *of the game Γ if*

$$a_{ij^*} \leq a_{i^*j^*} \leq a_{i^*j} \ , \qquad \begin{array}{c} 1 \leq i \leq m \\ 1 \leq j \leq n \ . \end{array}$$

In this case the number $v := a_{i^*j^*}$ is called the **value of game** Γ.

Lemma 1.2. *The game Γ with matrix A has a saddle point if and only if $\underline{v} = \overline{v}$.*

If matrix A has a saddle point the question for a solution of game Γ with A is trivial. None of the players is interested selecting another strategy than its optimal one.
A matrix game may have more than one saddle point. However:

Theorem 1.3 (Rectangular property). *Let (i^*, j^*) and (i^0, j^0) be saddle points of a matrix game Γ. Then (i^*, j^0) and (i^0, j^*) are its saddle points also, and the values at all saddle points are equal, i.e.*

$$a_{i^*j^*} = a_{i^*j^0} = a_{i^0j^*} = a_{i^0j^0} \ .$$

1.3 Mixed strategies

When a matrix game has no saddle point, i.e. if $\underline{v} < \overline{v}$ like in the "scissors–paper–stone"–game, we cannot solve the game in the sence of the last section.

Let us consider the game with

$$A = \begin{pmatrix} 5 & 1 \\ 3 & 4 \end{pmatrix} .$$

Here $\quad \underline{v} = \max_i \min_j \; a_{ij} = 3, \; \overline{v} = \min_j \max_i \; a_{ij} = 4 \; .$

Player P_1 can be sure to gain at least 3 playing strategy row $\hat{\imath} = 2$, player P_2 can guarantee that his loss is at most 4 playing strategy column $\hat{\jmath} = 2$. Nevertheless, situation $(2, 2)$ cannot be regarded as optimal behaviour. Suppose, player P_1 will follow his maxmin–strategy $\hat{\imath} = 2$. Then player P_2, if he guesses player P_1's decision, will choose column 1 instead of column 2. Then his loss will be 3 (instead of 4). P_1 may duplicate player P_2's considerations and decides to choose row 1 in order to receive 5. P_2 in his turn copying P_1's considerations will conclude, that it should be better for him to choose column 2 (his loss will be only 1), and so on.

Each player will make efforts to prevent his opponent from finding out his actual choise of strategy.

This uncertainty is caused by the fact that $\overline{v} - \underline{v} > 0$. The opponents could not agree how to share this amount. A way out of the situation is randomization: Players should choose their decision by random. Player P_1 should make use of some chance device to determine which strategy he is going to choose, similarly, player P_2 will also decide his choice of strategy by some chance method.

For instance, if P_1 chooses row 1 with probability 1/4 and row 2 with probability 3/4 his expected payoff will be $\frac{1}{4} \cdot 5 + \frac{3}{4} \cdot 3 = \frac{14}{4} > 3$, if P_2 chooses the first column, respectively, $\frac{1}{4} \cdot 1 + \frac{3}{4} \cdot 4 = \frac{13}{4} > 3$, if P_2 chooses the second column.

Definition 1.3. *Let be given the game $\Gamma = \langle S_1, S_2, H \rangle$ in pure strategies with matrix A (P_1 chooses $i \in S_1 = \{1, 2, \dots, m\}$, P_2 chooses $j \in S_2 = \{1, 2, \dots, n\}$, P_1 gains $H(i, j) = a_{ij}$ from P_2).*

The game $\hat{\Gamma} = \langle \hat{S}_1, \hat{S}_2, \hat{H} \rangle$ is called the **mixed extension of** Γ, *if*

$$\begin{aligned} \hat{S}_1 &= \{x \in \mathbb{R}^m \; : \; x \geq 0, \; \langle e, x \rangle = 1 \}, \\ \hat{S}_2 &= \{y \in \mathbb{R}^n \; : \; y \geq 0, \; \langle e, y \rangle = 1 \}, \\ \hat{H}(x, y) &= \langle x, Ay \rangle = \sum_{j=1}^{n} \sum_{i=1}^{m} \; a_{ij} x_i y_j, \quad e = [1, 1, \dots, 1]. \end{aligned}$$

In $\hat{\Gamma}$ player P_1 chooses a vector $x \in \hat{S}_1$, that is, he chooses his pure strategy i with probability x_i, $i = 1, 2, \dots, m$. Player P_2 chooses his pure strategy j with probability y_j, $j = 1, 2, \dots, n$. P_1 gains in $\hat{\Gamma}$ the expected value $\langle x, Ay \rangle$ from P_2.

Remarks:

1. $x \in \hat{S}_1$ is called a **mixed strategy of $\mathbf{P_1}$**.
   $y \in \hat{S}_2$ is called a **mixed strategy of $\mathbf{P_2}$**.

2. The strategy sets $\hat{S}_1$ and $\hat{S}_2$ are convex, closed, bounded sets. The payoff–function $\hat{H}(x, y)$ is a continuous function.

3. In contrast to game Γ the players have in $\hat{\Gamma}$ a continuum of strategies.

4. $\hat{\Gamma}$ is again a zero–sum game.

5. We assume the same rules for the behaviour of the players as we made at the beginning of section 1.1 (P_i takes its choice independently, both players have full information, are both interested in maximizing its own payoff, they act cautious).

Repeating considerations from sections 1.1 and 1.2 we will obtain analogous results to lemma 1.1, 1.2 and theorem 1.3. Lets follow this way.

If P_1 uses the strategy $x \in \hat{S}_1$, then his expected payoff is at least

$$\min\{\langle x, Ay\rangle \ : \ y \in \hat{S}_2\} \ . \tag{$\circ$}$$

Therefore, P_1 can choose $x \in \hat{S}_1$ so as to make ($\circ$) a maximum, i.e. he can be sure of an expected payoff not less than

$$\underline{\hat{v}} = \max_{x \in \hat{S}_1} \min_{y \in \hat{S}_2} \langle x, Ay\rangle \ . \tag{$\circ\circ$}$$

If player P_2 chooses the strategy $y \in \hat{S}_2$, then the expected payoff of player P_1 is at most

$$\max\{\langle x, Ay\rangle \ : \ x \in \hat{S}_1\} \ . \tag{$+$}$$

Player P_2 can choose $y \in \hat{S}_2$ so as to make (+) a minimum, i.e. he can prevent player P_1 from gaining an expected payoff greater than

$$\overline{\hat{v}} = \min_{y \in \hat{S}_2} \max_{x \in \hat{S}_1} \langle x, Ay\rangle \ . \tag{$++$}$$

We remark, that the outer extrema in ($\circ\circ$) resp. (++) are attained:
Since $\hat{S}_2$ is an convex, compact set, $\varphi(x) = \min\{\langle x, Ay\rangle \ : \ y \in \hat{S}_2\}$ is a continuous function on the closed set $\hat{S}_1$. The similar is true for $\psi(y) = \max\{\langle x, Ay\rangle \ : \ x \in \hat{S}_1\}$.

Lemma 1.4. *(analogue to lemma 1.1) For every matrix A in game $\hat{\Gamma}$ we have*

$$\underline{\hat{v}} \leq \overline{\hat{v}} \ .$$

Definition 1.4. *A tupel* $[x^*, y^*] \in \hat{S}_1 \times \hat{S}_2$ *is called a* **saddle point of function $\hat{\mathbf{H}}(\mathbf{x}, \mathbf{y})$**, *if*

$$\begin{cases} \langle x^*, Ay^* \rangle \geq \langle x, Ay^* \rangle \ , & \forall x \in \hat{S}_1 \\ \langle x^*, Ay^* \rangle \leq \langle x^*, Ay \rangle \ , & \forall y \in \hat{S}_2 \end{cases}$$

(i.e., $f(x) := \hat{H}(x, y^*)$ *has a global maximum at* x^*, $g(y) := \hat{H}(x^*, y)$ *has a global minimum at* y^**).*

Lemma 1.5. *(analogue to lemma 1.2)*
(i) The function $\hat{H}(x, y)$ *from game* $\hat{\Gamma}$ *has a saddle point if and only if*

$$\underline{\hat{v}} = \overline{\hat{v}} \qquad (*)$$

(ii) In the case if $(*)$ *is valid, we have:*
There exists a pair $[x^*, y^*] \in \hat{S}_1 \times \hat{S}_2$*, such that*

$$\begin{aligned} \underline{\hat{v}} &= \min_{y \in \hat{S}_2} \langle x^*, Ay \rangle = \langle x^*, Ay^* \rangle \\ &= \max_{x \in \hat{S}_1} \langle x, Ay^* \rangle = \overline{\hat{v}} \ . \end{aligned}$$

Definition 1.5. *If* $[x^*, y^*]$ *is a saddle point of* $\hat{H}(x, y)$ *in game* $\hat{\Gamma}$*, then we call* x^* *an* **optimal strategy of $\mathbf{P_1}$** *in* $\hat{\Gamma}$, y^* *an* **optimal strategy of $\mathbf{P_2}$** *in* $\hat{\Gamma}$ *and* $\hat{v} = \langle x^*, Ay^* \rangle$ *the* **value of game $\hat{\mathbf{\Gamma}}$**.

Remarks:

1. $\hat{\Gamma}$ is really an extension of the game Γ :
   - The pure strategies i resp. j in Γ are present in $\hat{\Gamma}$ as $x := e^i = [0, \ldots, 1, 0, \ldots, 0] \in \mathbb{R}^m$ resp. $y := e^j = [0, \ldots, 0, 1, 0, \ldots, 0] \in \mathbb{R}^n$ (e^i is the i'th unit vector of $\mathbb{R}^m$, e^j is the j'th unit vector of $\mathbb{R}^n$).
   - If $[i^*, j^*]$ is a saddle point of matrix A, then $[x^*, y^*]$ with $x^* := e^{i^*}$, $y^* := e^{j^*}$ forms a saddle point of $\hat{H}(x, y)$.
2. From the practical point of view there is some difference between Γ and $\hat{\Gamma}$:
   Suppose, that $\underline{v} < \overline{v}$ and $\underline{\hat{v}} = \overline{\hat{v}}$. If P_1 keeps to his pure maxmin–strategy in Γ, so he may be sure to gain at least the amount $\underline{v}$ (even if P_2 plays a mixed strategy). If P_1 plays his optimal mixed strategy x^* in $\hat{\Gamma}$ (i.e. his mixed maxmin–strategy in $\hat{\Gamma}$), then he will gain the value $\underline{\hat{v}} \geq \underline{v}$ only as an expected value. In a concrete realization of game $\hat{\Gamma}$ player P_1 realizes his optimal strategy x^*, i.e. he chooses a row i_1 and P_2 playing y^* selects some column j_1. Therefore, in this single game (realization of $\hat{\Gamma}$) player P_1 gains the payoff $a_{i_1 j_1}$, which may be even less than $\underline{v}$. That's why some experts do not accept mixed strategies.

Theorem 1.6. *(analogue to theorem 1.3) We consider game* $\hat{\Gamma} = \langle \hat{S}_1, \hat{S}_2, \hat{H} \rangle$*. If* $[x^*, y^*]$ *and* $[x^0, y^0]$ *are both saddle points of* $\hat{H}$ *in* $\hat{\Gamma}$*, then*

(i) $[x^*, y^0]$ *and* $[x^0, y^*]$ *are saddle points of* $\hat{H}$ *also,*

(ii) $\hat{H}(x^*, y^*) = \hat{H}(x^*, y^0) = \hat{H}(x^0, y^*) = \hat{H}(x^0, y^0)$
(i.e. the expected payoff does not depend on which of the optimal strategies the players realize).

1.4 Solvability in mixed strategies

In contrast to game Γ in the game $\hat{\Gamma}$ a saddle point always exists. This fundamental fact was first proved in 1928 by J.v.Neumann and this year is often meant as year of birth of the game theory.

Lemma 1.7. *We put for* $z \in \mathbb{R}^1$

$$\begin{aligned} M(z) &= \{\, x \in \mathbb{R}^m : \langle x, Ay \rangle \geq z,\ \forall y \in \hat{S}_2 \}, \\ N(z) &= \{\, x \in \mathbb{R}^m : \langle x, Ae^j \rangle \geq z,\ \forall y = 1, 2, \ldots, n \}, \end{aligned}$$

where e^j *stands for the* j*'th unit vector of* $\mathbb{R}^n$.
The following is valid:

$$M(z) = N(z),\ \forall z \in \mathbb{R}^1$$

Theorem 1.8. *The game* $\hat{\Gamma}$ *(game with matrix* A *in mixed strategies) is always solvable, i.e. function* $\hat{H}(x, y)$ *always has a saddle point and game* $\hat{\Gamma}$ *has a value.*

Corollary 1.9.

1. *If* $[z^*, x^*]$ *is an optimal solution of the linear progamming problem*

$$(P_1) \quad \begin{cases} z \to \max \\ \langle e, x \rangle = 1 \\ \langle x, Ae^j \rangle \geq z,\ \forall j = 1, 2, \ldots, n \\ x \geq 0 \end{cases},$$

   then x^* *is an optimal strategy of player* P_1 *in* $\hat{\Gamma}$ *and* z^* *is the value of game* $\hat{\Gamma}$ *and vice versa.*

2. *If* $[t^*, y^*]$ *is an optimal solution of the linear programming problem* (P_2)

$$(P_2) \quad \begin{cases} t \to \min \\ \langle e, y \rangle = 1 \\ \langle e^i, Ay \rangle \leq t,\ \forall i = 1, 2, \ldots, m \\ y \geq 0 \end{cases},$$

   which is dual to (P_1)*, then* y^* *is an optimal strategy of player* P_2 *in* $\hat{\Gamma}$ *and* t^* *is the value of the game* $\hat{\Gamma}$ *and vice versa.*

3. *If* $[z^*, x^*]$ *and* $[t^*, y^*]$ *are optimal solutions of* (P_1) *resp.* (P_2)*, then*

$$\begin{cases} z^* = t^*, \\ [x^*, y^*] \text{ is a saddle point of } \hat{H}(x, y) = \langle x, Ay \rangle, \\ (\langle e^i, Ay^* \rangle - t^*) \cdot x_i^* = 0,\ \forall i = 1, 2, \ldots, m \qquad (\Box) \\ (\langle x^*, Ae^j \rangle - z^*) \cdot y_j^* = 0,\ \forall j = 1, 2, \ldots, n \qquad (\Box\Box) \end{cases}$$

4. *If $[z^*, x^*]$ and $[t^*, y^*]$ are feasible solutions of LOP (P_1) resp. (P_2) and $(\Box)$, $(\Box\Box)$ are satisfied, then $[x^*, y^*]$ is a saddle point of $\hat{H}$ in $\hat{\Gamma}$ and vice versa.*

5. *If $[z^*, x^*]$ and $[t^*, y^*]$ are feasible solutions of the LOP (P_1) resp. (P_2) and $z^* = t^*$, then $[x^*, y^*]$ is a saddle point of $\hat{H}$ in $\hat{\Gamma}$ and $z^* = t^*$ is the value of game $\hat{\Gamma}$ and vice versa.*

Example:
Lets again consider the "scissors–paper–stone" game from section 0.1

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} .$$

Because of the symmetry of A we may expect, that

$$x^* = [1/3, 1/3, 1/3],\ y^* = [1/3, 1/3, 1/3] \text{ and } v = 0$$

form the solution of game $\hat{\Gamma}$ (play with A in mixed strategies). It is easy to verify that $[z^*, x^*]$ with $z^* = 0$ is a feasible solution of (P_1)

$$(P_1) \begin{cases} z \to \max \\ x_1 + x_2 + x_3 = 1 \\ -x_2 + x_3 \geq z \\ x_1 - x_3 \geq z \\ -x_1 + x_2 \geq z \\ x_i \geq 0,\ i = 1, 2, 3 \,. \end{cases}$$

$[t^*, y^*]$ with $t^* = 0$ forms a feasible solution of (P_2)

$$(P_2) \begin{cases} t \to \min \\ y_1 + y_2 + y_3 = 1 \\ y_2 - y_3 \leq t \\ -y_1 + y_3 \leq t \\ y_1 - y_2 \leq t \\ y_j \geq 0,\ j = 1, 2, 3 \,. \end{cases}$$

Since $t^* = z^*$ we conclude from Corollary 1.9, that $[x^*, y^*]$ is a saddle point in $\hat{\Gamma}$, i.e. $[1/3, 1/3, 1/3]$ is an optimal strategy of both players in the "scissors–paper–stone" game.

1.5 Optimal strategies and their properties

Definition 1.6. *We say that the pure strategy $i \in \{1, 2, \ldots, m\}$ belongs to the* **spectrum** *of player P_1, if there exists an optimal strategy $x^* \in \hat{S}_1$ in game $\hat{\Gamma}$ with $x_i^* > 0$ (i.e. the pure strategy i is played in the mixed strategy x^* with a positive probability).*

Lemma 1.10.
If i_0 belongs to the spectrum of the optimal strategy x^* of player P_1 in game $\hat{\Gamma}$, then $\langle e^{i_0}, Ay^* \rangle = \hat{v}$, $\forall y^*$ which are optimal strategies of P_2 in $\hat{\Gamma}$

(i.e. if P_2 plays an optimal strategy then P_1 can play any of his spectrum strategies in order to gain the value $\hat{v}$ of the game (as expected value)).

Remarks:
1. The lemma does not state that e_0^i is an optimal strategy of P_1.
2. An analogous lemma is true for the spectrum of player P_2.

Lemma 1.11. *Let be given $k > 0$, $\alpha \in \mathbb{R}^1$. We put*

$$\tilde{a}_{ij} = k \cdot a_{ij} + \alpha, \ \forall i, j,$$
$$\tilde{A} = \{\tilde{a}_{ij}\}_{i=1,j=1}^{m,n}, \ A = \{a_{ij}\}_{i=1,j=1}^{m,n}.$$

Then the following is valid:
$x^* \in \hat{S}_1$, $y^* \in \hat{S}_2$ are optimal strategies in the game $\hat{\Gamma}$ with matrix A and $\hat{v}$ is the value of $\hat{\Gamma}$ if and only if x^*, y^* are optimal strategies in the game with matrix $\tilde{A}$ and $v = k \cdot \hat{v} + \alpha$ is the value of the play with matrix $\tilde{A}$.

Remark:
As a consequence of lemma 1.11 we always may assume, that the value $\hat{v}$ of $\hat{\Gamma}$ is equal to zero.

Lemma 1.12. *Let i be a pure strategy of player P_1 in the game Γ. Then exactly one of the two alternatives is true:*

(a) *i belongs to the spectrum of P_1*
(i.e., there exists an optimal strategy x^* of P_1 in $\hat{\Gamma}$ with $x_i^* > 0$).

(b) *Player P_2 has an optimal strategy y^* in the game $\hat{\Gamma}$ which satisfies the condition $\langle e^i, Ay^* \rangle < \hat{v}$, where $\hat{v}$ denotes the value of $\hat{\Gamma}$.*

Definition 1.7. *Let $\hat{x}$ and $\bar{x}$ be mixed strategies of P_1 in game $\hat{\Gamma}$ with matrix A. We say that $\hat{x}$* **strongly dominates** *over $\bar{x}$, if*

$$\langle \hat{x}, Ae^j \rangle > \langle \bar{x}, Ae^j \rangle, \ \forall j = 1, 2, \ldots, n.$$

Analogously we say for mixed strategies $\hat{y}$ and $\bar{y}$ of P_2 that $\hat{y}$ strongly dominates over $\bar{y}$, if

$$\langle e^i, A\hat{y} \rangle < \langle e^i, A\bar{y} \rangle, \ \forall j = 1, 2, \ldots, m.$$

Remarks:
(1) It is easy to prove that $\hat{x}$ strongly dominates over $\bar{x} \Leftrightarrow$
$\langle \hat{x}, Ay \rangle > \langle \bar{x}, Ay \rangle$, $\forall y \in \hat{S}_2$.

(2) If the pure strategy $\hat{x} = e^{i_0}$ strongly dominates over the pure strategy $\bar{x} = e^{i_1}$, then $a_{i_0 j} > a_{i_1, j}$, $\forall j$
(the elements of the i_0th row are all greater than the corresponding elements in the i_1th row of the matrix A).

Lemma 1.13. *None of the strongly dominated pure strategies can belong to the spectrum of a player.*

Corollary 1.14.
The lemma offers a possibility to reduce the size of the matrix A:

If the pure strategy e^{i_0} of player P_1 is strongly dominated by a strategy $\hat{x}$, then $x^*_{i_0} = 0$ in any optimal strategy x^* of player P_1. Therefore, if we are looking for optimal strategies we may omit the i_0th row from matrix A.

Definition 1.8. *If A is a skew–symmetric matrix (i.e. $A = -A^\top$), then we call the game with matrix A a* **symmetric game**.

Lemma 1.15. *Let $\hat{\Gamma}$ be a symmetric game. Then*

(*i*) *The value $\hat{v}$ of $\hat{\Gamma}$ is equal to zero.*

(*ii*) *$x^* \in \hat{S}_1$ is an optimal strategy of player P_1 in $\hat{\Gamma}$ if and only if $y^* := x^*$ is an optimal strategy of player P_2 in $\hat{\Gamma}$.*

Chapter 2

Games played over the unit square

2.1 Games in normal (strategic) form

2.1.1 n–person games (noncooperative)

Definition 2.1. *Let us be given*

$$\begin{cases} n \text{ persones } P_1, P_2, \ldots, P_n \\ n \text{ sets } S_i \subset \mathbb{R}^{n_i},\ i = 1, 2, \ldots, n, \text{ and} \\ n \text{ functions } H_i \ :\ \prod_{i=1}^{n} S_i \to \mathbb{R} \end{cases} .$$

The tupel

$$\Gamma = \langle P_1, P_2, \ldots, P_n, S_1, S_2, \ldots, S_n, H_1, H_2, \ldots, H_n \rangle$$

is called a **n–person game in normal form** *if it is understood that P_i chooses independently of the other P_j, $j \neq i$, an element $\sigma^i \in S_i$ and after the whole tupel $(\sigma^1, \sigma^2, \ldots, \sigma^n)$ is chosen the player P_i gains a payoff $H_i(\sigma^1, \sigma^2, \ldots, \sigma^n)$, $i = 1, 2, \ldots, n$.*

Remarks

1. The rules of the game include that all players are well informed, i.e. P_i knows not only his own S_i and H_i but also the sets S_j and functions H_j of all other players. In addition we assume that each of the players is interested to maximize his own payoff H_i.

2. We will use the following notations:
   P_i – i-th player
   S_i – strategy set of P_i
   H_i – payoff–function of P_i
   $\sigma = [\sigma^1, \sigma^2, \ldots, \sigma^n] \in \prod_{i=1}^{n} S_i$ – situation in the game Γ.

Definition 2.2. *A situation* $\sigma = [\sigma^1, \sigma^2, \ldots, \sigma^n]$ *in the game* Γ *is called* **acceptable to $\mathbf{P_i}$**, *if*

$$H_i(\sigma^1, \sigma^2, \ldots, \sigma^{i-1}, s, \sigma^{i+1}, \ldots, \sigma^n) \le H_i(\sigma),\ \forall s \in S_i.$$

Definition 2.3. *A situation* $\sigma = [\sigma^1, \sigma^2, \ldots, \sigma^n]$ *in the game* Γ *is called a (Nash-)***equilibrium** *if it is acceptable to all players* $P_1, P_2, \ldots, P_n$.

In matrix games we have had $n = 2$, $H_1 = \langle x, Ay \rangle$, $H_2 = -\langle x, Ay \rangle$ and therefore $[x^*, y^*] \in S_1 \times S_2$ is an equilibrium if

$$\begin{cases} \langle x, Ay^* \rangle \le \langle x^*, Ay^* \rangle,\ \forall x \in S_1 \\ -\langle x^*, Ay \rangle \le -\langle x^*, Ay^* \rangle,\ \forall y \in S_2, \end{cases}$$

i.e. if $[x^*, y^*]$ is a saddle point of $\langle x, Ay \rangle$.

Definition 2.4. *We consider two n–person games in normal form*

$$\Gamma = \langle P_1, P_2, \ldots, P_n, S_1, S_2, \ldots, S_n, H_1, H_2, \ldots, H_n \rangle$$

and

$$\hat{\Gamma} = \langle P_1, P_2, \ldots, P_n, S_1, S_2, \ldots, S_n, \hat{H}_1, \hat{H}_2, \ldots, \hat{H}_n \rangle.$$

We say that Γ *and* $\hat{\Gamma}$ *are* **strategically equivalent** *if there exist constants* $k > 0$, *and* $c_1, c_2, \ldots, c_n$ *such that for every* $\sigma \in \prod_{i=1}^{n} S_i$

$$H_i(\sigma) = k \cdot \hat{H}_i(\sigma) + c_i,\ \forall i = 1, 2, \ldots, n.$$

We will use the notation $\Gamma \sim \hat{\Gamma}$.

Lemma 2.1. *The strategical equivalence is reflexive, symmetrical and transitive.*

Theorem 2.2. *(generalization of theorem 1.12)*
Strategically equivalent games have the same equilibria.

Definition 2.5. *We call the n–person game in normal form*

$$\Gamma = \langle P_1, P_2, \ldots, P_n, S_1, S_2, \ldots, S_n, H_1, H_2, \ldots, H_n \rangle$$

a **constant–sum game** *if there exists a* $c \in \mathbb{R}^1$ *such, that*

$$\sum_{i=1}^{n} H_i(\sigma) = c,\ \forall \sigma \in \prod_{i=1}^{n} S_i\ .$$

In the case if $c = 0$ *we say that* Γ *is a* **zero–sum game**.

Theorem 2.3. *Let the n–person game in normal form* Γ *be a constant–sum game. Then* Γ *is strategically equivalent to a zero–sum game.*

2.1.2 Antagonistic games

Definition 2.6. *Two–person zero–sum games in normal form we call* **antagonistic games***.*

Since in such games $H_2(\sigma) = -H_1(\sigma)$ *we will use the notation* $\Gamma = \langle P_1, P_2, S_1, S_2, H\rangle$ *instead of* $\Gamma = \langle P_1, P_2, S_1, S_2, H_1, H_2\rangle$.

Lemma 2.4. *(generalization of lemma 1.1 and lemma 1.4)*
Let $S_1 \subset \mathbb{R}^m$, $S_2 \subset \mathbb{R}^n$, $H : S_1 \times S_2 \to \mathbb{R}$. *Then always the inequality*

$$\underline{v} \stackrel{\text{def}}{=} \sup_{x\in S_1} \inf_{y\in S_2} H(x,y) \le \inf_{y\in S_2} \sup_{x\in S_1} H(x,y) \stackrel{\text{def}}{=} \overline{v}$$

is valid.

Theorem 2.5. *(generalization of lemma 1.2 and lemma 1.5)*
Let $\Gamma = \langle P_1, P_2, S_1, S_2, H\rangle$ *be an antagonistic game. In* Γ *an equilibrium exists if and only if*

(*i*) *in both* $\sup_{x\in S_1} \inf_{y\in S_2} H(x,y)$ *and* $\inf_{y\in S_2} \sup_{x\in S_1} H(x,y)$ *the outer extrema are attained,*

(*ii*) $\max_{x\in S_1} \inf_{y\in S_2} H(x,y) = \min_{y\in S_2} \sup_{x\in S_1} H(x,y)$.

Corollary 2.6. *(rectangular property, cf. theorem 1.3 and theorem 1.6)*
Suppose that $[x^*, y^*]$ *and* $[x^0, y^0]$ *are both equilibria in the antagonistic game* $\Gamma = \langle P_1, P_2, S_1, S_2, H\rangle$. *Then* $[x^*, y^0]$ *and* $[x^0, y^*]$ *are equilibria too and we have* $H(x^*, y^*) = H(x^0, y^*) = H(x^*, y^0) = H(x^0, y^0)$.

Definition 2.7. *Let* Γ *be an antagonistic game and* $[x^*, y^*]$ *an equilibrium in* Γ. *The number* $v = H(x^*, y^*)$ *is called the* **value of the game** $\mathbf{\Gamma}$ *and the vectors* x^* *and* y^* *are called an* **optimal strategy** *of* P_1 *respectively of* P_2 *in* Γ.

Definition 2.8. *We consider two antagonistic games* $\Gamma = \langle P_1, P_2, S_1, S_2, H\rangle$ *and* $\Gamma' = \langle P_1', P_2', S_1', S_2', H'\rangle$. *We say that* Γ *and* Γ' *are* **continuously isomorphic** *if there exist two continuous one–to–one mappings*

$$\varphi : S_1 \to S_1', \quad \psi : S_2 \to S_2'$$

such, that

$$H'(\varphi(x), \psi(y)) = H(x,y), \quad \forall x \in S_1,\ \forall y \in S_2.$$

Lemma 2.7. *Let* Γ *and* Γ' *be continuously antagonistic games. Then the following is true:*

(*i*) *An equilibrium exists in* Γ' *if and only if there exists an equilibrium in* Γ.

(*ii*) *If* x^* *and* y^* *are optimal strategies of* P_1 *resp.* P_2 *in* Γ, *then* $\varphi(x^*)$ *and* $\psi(y^*)$ *are optimal strategies of* P_1' *resp.* P_2' *in* Γ'.

Concluding remark: Note that during section 2.1 we have not made any assumptions like convexity, connectivity, continuity, number of elements of S_i resp. H_i.

2.2 Games over the unit square

2.2.0 The subject

Matrix games are typical representatives of a finite antagonistic game. Games played over the unit square are the representatives of an infinite antagonistic game.

Definition 2.9. *Let* $\Gamma = \langle P_1, P_2, S_1, S_2, H\rangle$ *be an antagonistic game. If*
$S_1 = S_2 = [0,1] \stackrel{\text{def}}{=} \{x \in \mathbb{R} \,:\, 0 \leq x \leq 1\}$,
then we call Γ *a* **game over the unit square**.

The following example shows that in such games we, generally, can not expect that there exists an equilibrium $[x^*, y^*]$ in pure strategies (i.e. $x^* \in S_1,\ y^* \in S_2$).

Example: $H(x,y) = (x-y)^2$ (payoff in a persuit game)

For any $x \in [0,1]$ we have $\min\limits_{0\leq y\leq 1} H(x,y) = 0$ and therefore

$$\underline{v} = \max_{0\leq x\leq 1}\min_{0\leq y\leq 1} H(x,y) = 0.$$

Since $\varphi(y) = \max\limits_{0\leq x\leq 1}(x-y)^2$ is the maximum of a convex function we have

$$\begin{aligned}\varphi(y) &= \max\{(x-y)^2|_{x=0},\ (x-y)^2|_{x=1}\}\\ &= \max\{y^2, (\ (1-y)^2\}.\end{aligned}$$

Thus we get $\overline{v} = \min\limits_{0\leq y\leq 1}\max\limits_{0\leq x\leq 1} H(x,y) = \frac{1}{4}$. Because $\underline{v} \neq \overline{v}$ we conclude from theorem 2.5 that an equilibrium does not exist.

2.2.1 Mixed extension

We will be concerned with the antagonistic game $\Gamma = \langle P_1, P_2, S_1, S_2, H\rangle$, where $S_1 = S_2 = [0,1]$ and $H : [0,1] \times [0,1] \to \mathbb{R}$.

As we have seen such a game may not have any equilibria. The situation is analogous to the case of finite games. There we allowed randomization of strategies. Here we will act on the same way.

Let F and G be probability distributions on the unit interval $[0,1]$: $F, G : [0,1] \to \mathbb{R}$.
Then, if player P_1 chooses a distribution function F he has committed himself to pick a point in the interval $[0,x)$ with the probability $F(x)$.

Define the strategy set $\hat{S}_1$ of player P_1 as the family of all distribution functions on $[0,1]$. Analogously we will act for P_2. He will choose a distribution function $G \in \hat{S}_2 := \hat{S}_1$. We define the payoff function $\hat{H} : \hat{S}_1 \times \hat{S}_2 \to \mathbb{R}$ of player P_1 by the Stiltjes integral

(∘) $\hat{H}(F,G) = \int\limits_0^1 \int\limits_0^1 H(x,y)dF\,dG.$

Expression (∘) is the expected payoff for player P_1 if (pure) strategies x and y are chosen according to the probability distributions F resp. G.

Definition 2.10. *The antagonistic game* $\hat{\Gamma} = \langle P_1, P_2, \hat{S}_1, \hat{S}_2, \hat{H} \rangle$ *thus defined is called the* **mixed extension** *of the game* $\Gamma = \langle P_1, P_2, S_1, S_2, H \rangle$ *played over the unit square.*

Remarks:

1. Since we will primarily be interested in mixed extensions, we will use the term "game over the unit square" for $\hat{\Gamma}$ too.
2. Appendix A gives some facts conected with disribution functions. Here we will mention only the following properties of $F \in \hat{S}_1$:

$$\begin{cases} F(x) \geq 0, \ \forall x \in [0,1] \\ F(0) = 0, \\ F(1) = 1, \\ x < x' \ \Rightarrow \ F(x) \leq F(x') \\ \text{if } x \in (0,1), \text{ then } F(x) = \lim\limits_{\varepsilon \to 0} F(x+\varepsilon) \end{cases}$$

3. We remark that $\hat{H}(F,G)$ always exists if H is a continuous function over $S_1 \times S_2$.
4. The game Γ in pure strategies is embedded in the mixed extension $\hat{\Gamma}$:

   With the pure strategy $a \in [0,1]$ we can associate the distribution function

$$I_a(x) = \begin{cases} 0 & , \quad 0 \leq x < a \\ 1 & , \quad x \geq a \end{cases} \qquad \text{if } a > 0,$$

$$I_a(x) = \begin{cases} 0 & , \quad x = 0 \\ 1 & , \quad x > 0 \end{cases} \qquad \text{if } a = 0.$$

   Setting $F := I_a$, $G := I_b$ we will receive

$$\hat{H}(F,G) = \int\limits_0^1 \int\limits_0^1 H(x,y)\, dI_a(x)\, dI_b(y) = H(a,b).$$

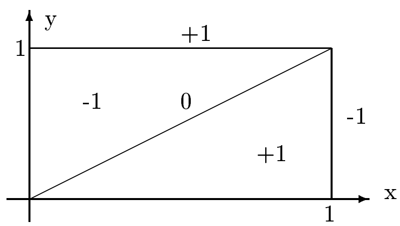
5. Since we introduced $\hat{\Gamma}$ as an antagonistic game, all termes defined in section 2.1 and all propositions made there are applicable to $\hat{\Gamma}$.

Theorem 2.8. *Let* $\Gamma = \langle P_1, P_2, S_1, S_2, H \rangle$ *be a game on the unit square. If* H *is continuous on* $[0,1] \times [0,1]$*, then in the mixed extension* $\hat{\Gamma}$ *an equilibrium exists.*

Our proof allows to apply linear programming software in order to get an approximate solution of $\hat{\Gamma}$.

Consider the payoff–function

$$H(x,y) = \begin{cases} 1 & , \quad y < x < 1 \\ -1 & , \quad x < y < 1 \\ 0 & , \quad 0 \le x = y \le 1 \\ -1 & , \quad x = 1,\ y \in [0,1) \\ 1 & , \quad y = 1,\ x \in [0,1) \end{cases} \qquad (+)$$



H is the payoff in the following game: Both players choose a number x resp. y from the unit interval $[0,1]$ simultaneously. The one who has chosen the bigger number gains the payoff $+1$.

Lemma 2.9. *In the game on the unit square with payoff–function H according to $(+)$ an equilibrium does not exist even in the mixed extension.*

From the definition of an equilibrium and from theorem 2.5 we get that the following propositions $(\circ)$ and $(\circ\circ)$ are equivalent:

$(\circ)$ $\quad [F^*, G^*] \in \hat{S}_1 \times \hat{S}_2$ is an equilibrium in game $\hat{\Gamma}$

$$(\circ\circ) \quad \begin{cases} [F^*, G^*] \in \hat{S}_1 \times \hat{S}_2 \text{ and } v \in \mathbb{R} \text{ satisfy the conditions} \\ \qquad \begin{cases} \inf\limits_G \hat{H}(F^*, G) = v \\ \sup\limits_F \hat{H}(F, G^*) = v \end{cases} \end{cases}$$

Example: Prove that $F^*(x) = I_{1/2}(x)$, $G^*(y) = \frac{1}{2}I_0(y) + \frac{1}{2}I_1(y)$ together with $v = 4/5$ determine a solution of the game on the unit square with the payoff–function

$$H(x,y) = (1 + (x-y)^2)^{-1}.$$

Finally we should mention that all properties outlined in sections 2.1.1 and 2.1.2 are applicable to games on the unit square.

2.2.2 Spectral points

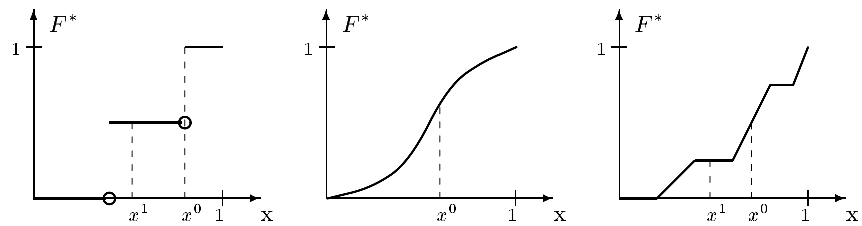
Definition 2.11. *Consider the mixed extension $\hat{\Gamma}$ of a game over the unit square. We say, that $x^0 \in [0,1]$ belongs to the* **spectrum** *of player P_1 in game $\hat{\Gamma}$, if there exists an optimal strategy F^* of P_1 such, that $\forall \varepsilon > 0$ the strong inequality*

$$\int_{x^0-\varepsilon}^{x^0+\varepsilon} dF^*(x) > 0$$

holds.

Remarks:

1.



the points x^0 are spectral points, x^1 are not

2. If $F^*(x)$ is constant in some neighboorhood of x^1, then x^1 not belongs to the spectrum.

3. If F^* is differentiable at x^0 and $\frac{dF^*(x^0)}{dx} > 0$, then x^0 is a spectral point.

Theorem 2.10.
Suppose that the mixed extension $\hat{\Gamma}$ of the game $\Gamma = \langle P_1, P_2, S_1, S_2, H\rangle$ over the unit square has an equilibrium. Then the following is true:

(i) *If for any $y \in [0,1]$ the function $H(\cdot, y)$ is continuous on $[0,1]$ and $x^0 \in [0,1]$ belongs to the spectrum of P_1, then for each optimal strategy G^* of player P_2 we have*

$$\hat{H}(I_{x^0}, G^*) = v$$

(v denotes the value of $\hat{\Gamma}$).

(ii) *The analogue for P_2 is also true .*

Further results (relating to P_1):

- If H is continuous on $[0,1] \times [0,1]$, then P_1 has at least one spectral point.
- If $\forall y \in [0,1]$ the function $H(\cdot, y)$ is strictly concave and continuous on $[0,1]$, then P_1 has exactly one spectral point.
- If $H(x,y) = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} u_i(x) \cdot v_j(y)$ with continuous functions u_i and v_j, then P_1 has at most m spectral points.
- Strongly dominated pure strategies can not belong to the spectrum.

2.2.3 Convex games over the unit square

Theorem 2.11. *We consider a game Γ over the unit square. Suppose, that $\forall x \in [0,1]$ the function $H(x, \cdot)$ is a continuous, convex function on $[0,1]$. Then for the mixed extension $\hat{\Gamma}$ of Γ we have:*

(i) *$\hat{\Gamma}$ has a value, i.e. there exists a number v such, that*

$$v = \inf_G \sup_F \hat{H}(F,G) = \sup_F \inf_G \hat{H}(F,G).$$

(ii) *Player* P_2 *has in* $\hat{\Gamma}$ *a pure optimal strategy, i.e. there exists an* $a \in [0,1]$ *such, that*

$$\hat{H}(F, I_a) \le v, \quad \forall F \in \hat{S}_1,$$

where

$$I_a(y) = \begin{cases} 0 & , \quad 0 \le y < a \\ 1 & , \quad a \le y \le 1 \end{cases} \qquad \text{if } a > 0,$$

$$I_a(y) = \begin{cases} 0 & , \quad y = 0 \\ 1 & , \quad y > 0 \end{cases} \qquad \text{if } a = 0.$$

(iii) *For every* $\varepsilon > 0$ *player* P_1 *has an* ε*–optimal strategy which is a convex linear combination of at most two pure strategies, i.e. for* $\varepsilon > 0$ *there exist* $c, d \in [0,1]$ *and* $\lambda \in [0,1]$ *such, that with*

$$F^*(x) = \lambda I_c(x) + (1-\lambda) I_d(x)$$

we have

$$\hat{H}(F^*, G) \ge v - \varepsilon, \quad \forall G \in \hat{S}_2.$$

Remarks:

1. An analogous result is true, if $H(\cdot, y)$ is continuous and concave in x for all fixed $y \in [0,1]$.
2. An analogous result can be proven for antagonistic games where $S_1 \subset \mathbb{R}^m$, $S_2 \subset \mathbb{R}^n$ are convex, compact sets. In this case the ε–optimal strategy of P_1 is a convex linear combination of at most $(m+1)$ pure strategies.
3. We note, that under the conditions of theorem 2.11 an equilibrium $[F^*, G^*] \in \hat{\Gamma}$ may not exist.

Corollary 2.12. *We consider an antagonistic game* $\Gamma = \langle P_1, P_2, S_1, S_2, H \rangle$ *over the unit square. Suppose that*

$$\begin{cases} \forall x \in [0,1] \text{ function } H(x, \cdot) \text{ is convex and} \\ \qquad\qquad \text{continuous on } [0,1], \\ \forall y \in [0,1] \text{ function } H(\cdot, y) \text{ is concave and} \\ \qquad\qquad \text{continuous on } [0,1]. \end{cases}$$

Then in game Γ *there exists an equilibrium, i.e. both* P_1 *and* P_2 *have optimal strategies.*

As we know from convex analysis, convex functions are directionally differentiable. For reason of practical (analytical) calculation of optimal strategies we will use this fact and prove a theorem similarly to theorem 2.11 but under the additional condition that the existence of an equilibrium in $\hat{\Gamma}$ is guaranteed (cf. theorem 2.8).

Theorem 2.13. *We consider the game* Γ *over the unit square with a continuous (on* $[0,1] \times [0,1]$*) function* H*. Additionally we suppose that for any fixed* $x \in [0,1]$

the function $H(x,\cdot)$ is convex on $[0,1]$. Then for the mixed extension $\hat{\Gamma}$ of Γ the following is valid:

(*i*) *The value v of $\hat{\Gamma}$ is equal to*

$$\min_{0\le y\le 1}\max_{0\le x\le 1} H(x,y) \qquad (\circ)$$

(*ii*) *P_2 has a pure optimal strategy I_{y^*} in the game $\hat{\Gamma}$, where as y^* may be chosen any y that realizes the outer minimum in $(\circ)$.*

(*iii*) *P_1 has an optimal strategy $F^* \in \hat{S}_1$, which satisfies the following conditions*

- *case, if $y^* = 0:$* $\quad F^*(x) = I_{x^*}(x)$, *where*

$$\begin{cases} x^* \in [0,1] \\ H(x^*,0) = v \\ \Psi'(0;+1) \ge 0. \end{cases}$$

Here and further on $\Psi(y)$ stands for

$$\Psi(y) := H(x^*,y).$$

- *case, if $y^* = 1:$* $\quad F^*(x) = I_{x^*}(x)$, *where*

$$\begin{cases} x^* \in [0,1] \\ H(x^*,1) = v \\ \Psi'(1;-1) \le 0. \end{cases}$$

- *case, if $0 < y^* < 1:$* $\quad F^*(x) = \alpha I_{x^*}(x) + (1-\alpha) I_{x^{**}}(x)$ *where*

$$\begin{cases} 0 \le \alpha \le 1 \\ x^*, x^{**} \in [0,1] \\ \Psi'(y^*;+1) \ge 0 \\ \hat{\Psi}'(y^*;+1) \le 0 \\ H(x^*,y^*) = v \\ H(x^{**},y^*) = v \\ \alpha\Psi'(y^*;1) + (1-\alpha)\hat{\Psi}'(y^*;1) = 0. \end{cases}$$

Here $\hat{\Psi}$ denotes the function $\hat{\Psi}(y) = H(x^{**},y)$.

Example 1: Using theorem 2.13 solve the game with $H(x,y) = (x-y)^2$ (cf. the example from section 2.2.0).

Example 2: Solve the game with $H(x,y) = -2x^2 + y^2 + 3xy - x - 2y$.

2.2.4 Games of timing

2.2.4.1 The conception of timing games

Definition 2.12. *We consider an antagonistic game* $\Gamma = \langle P_1, P_2, S_1, S_2, H\rangle$ *over the unit square. We call* Γ *a* **game of timing**, *if there exist functions*

$$\varphi : [0,1] \to \mathbb{R} \text{ and } L, K : [0,1]\times[0,1] \to \mathbb{R}$$

such, that

$$(a) \qquad H(x,y) = \begin{cases} K(x,y) & , \quad x < y \\ \varphi(x) & , \quad x = y \\ L(x,y) & , \quad x > y \end{cases}$$

$$(b) \qquad \begin{cases} K \text{ is continuous on } \{[x,y] \in \mathbb{R}^2 : x \le y,\ 0 \le x,\ y \le 1\} \\ L \text{ is continuous on } \{[x,y] \in \mathbb{R}^2 : y \le x,\ 0 \le x,\ y \le 1\} \\ \varphi \text{ is continuous on } [0,1] \end{cases}$$

(c) H *is not continuous on* $[0,1]\times[0,1]$.

As suggest theorem 2.8 and lemma 2.9 we cannot generally expect, that for such games an equilibrium exists.

2.2.4.2 Example

Solve the game from section 0.2.
(classical duel without a silencer)

2.2.4.3 A result about the structure of optimal strategies

Let us consider mixed strategies $F^* \in \hat{S}_1$ and $G^* \in \hat{S}_2$ that have properties (i) and (ii):

(i) $F^*(x)$ and $G^*(y)$ are differentiable on $[0,1]$

(ii) There exist numbers $b, c, d, e \in [0,1]$ with $b < c$ and $d < e$ such, that

$$(a) \qquad f^*(x) := \frac{dF^*(x)}{dx} = \begin{cases} > 0 & , \quad x \in (b,c) \\ = 0 & , \quad x \notin (b,c) \end{cases}$$

$$(b) \qquad g^*(y) := \frac{dG^*(y)}{dy} = \begin{cases} > 0 & , \quad y \in (d,e) \\ = 0 & , \quad y \notin (d,e) \end{cases} \quad .$$

Theorem 2.14. *We consider a game* Γ *of timing. Suppose that the mixed strategies* $F^* \in \hat{S}_1$ *and* $G^* \in \hat{S}_2$ *fulfill conditions* (i) *and* (ii) *from above. Then the following propositions* (1) *and* (2) *are equivalent:*

(1) $[F^*, G^*]$ *is an equilibrium in* $\hat{\Gamma}$.

(2) *There exists a number* v *such, that*

$$\begin{cases} \hat{H}(F^*, I_y) \ge v & , \quad \forall y \in [0,1] & (3) \\ \hat{H}(F^*, I_y) = v & , \quad \forall y \in (d,e) & (4) \\ \hat{H}(I_x, G^*) \le v & , \quad \forall x \in [0,1] & (5) \\ \hat{H}(I_x, G^*) = v & , \quad \forall x \in (b,c) & (6) \end{cases}$$

Corollary 2.15. *We consider again a game* Γ *of timing with payoff function*

$$H(x,y) = \begin{cases} K(x,y) & , \quad x < y \\ \varphi(x) & , \quad x = y \\ L(x,y) & , \quad y < x \end{cases}$$

where K *and* L *are supposed to be partially differentiable. Under the conditions, that*
– the assumption of theorem 2.14 is fulfilled, and
– that proposition (2) *of theorem 2.14 is valid*
we have

$\forall x \in (b,c)$ *holds*

$$\begin{aligned} [K(x,x) - L(x,x)]g^*(x) &= \int_0^x \frac{\partial L(x,y)}{\partial x} \cdot g^*(y)\,dy + \\ &+ \int_x^1 \frac{\partial K(x,y)}{\partial x} \cdot g^*(y)\,dy \end{aligned} \tag{7}$$

$\forall y \in (d,e)$ *holds*

$$\begin{aligned} [L(y,y) - K(y,y)]f^*(y) &= \int_0^y \frac{\partial K(x,y)}{\partial y} \cdot f^*(x)\,dx + \\ &+ \int_y^1 \frac{\partial L(x,y)}{\partial y} \cdot f^*(x)\,dx \end{aligned} \tag{8}$$

The corollary gives the opportunity to find the density of the distribution functions F^* resp. G^* as solutions of two integral equations.

Corollary 2.16.
Given a game Γ *of timing with partially differentiable functions* K *and* L*. If we managed to find a solution of the system of relations* $(*)$

$$(*) \qquad \begin{cases} (1) - (8) \\ 0 \le b < c \le 1, \ 0 \le d < e \le 1 \end{cases}$$

with the unknown quantities b, c, d, e, F^*, G^* *and* v*, then* F^* *and* G^* *are optimal strategies of* P_1 *resp.* P_2*.*

Definition 2.13. *Let* $\Gamma = \langle P_1, P_2, S_1, S_2, H \rangle$ *be a game over the unit square. We call* Γ *a* **symmetric game***, if*

$$H(x,y) = -H(y,x), \quad \forall x, y \in [0,1].$$

Remarks:

1. Obviously, a game of timing is symmetric, if

$$\begin{cases} L(x,y) = -K(y,x) & , \quad \forall [x,y] \in \{[x,y] \in \mathbb{R}^2 : x \geq y,\ 0 \leq x,y \leq 1\} \\ \varphi(x) = 0 & , \quad \forall x \in [0,1] \end{cases}$$

If a symmetric game over the unit square has an equilibrium, then

$$\begin{cases} v = 0 \text{ is its value} \\ P_1 \text{ and } P_2 \text{ have the same optimal strategies} \end{cases}$$

2. In a symmetric game of timing we can (under system $(*)$) assume, that $b = d,\ c = e$.

Corollary 2.17. *Consider a symmetric game Γ of timing with partially differentiable functions K and L. If we managed to prove the consistency of the system $(**)$ and find a solution part F^*, then $\hat{\Gamma}$ has an equilibrium and F^* is an optimal strategy of P_1 and P_2.*

$$(**) \qquad \begin{cases} \hat{H}(F^*, I_y) = 0,\ \forall y \in (b,c) \\ \quad (F^*(x))' = f^*(x) \\ f^*(x) = \begin{cases} > 0 & , \quad x \in (b,c) \\ = 0 & , \quad x \notin (b,c) \end{cases} \\ \hat{H}(F^*, I_y) \geq 0,\ \forall y \in [0,1] \\ [L(y,y) - K(y,y)] \cdot f^*(y) = \int\limits_b^y \frac{\partial K(x,y)}{\partial y} f^*(x)\, dx + \\ \qquad + \int\limits_y^c \frac{\partial L(x,y)}{\partial y} f^*(x)\, dx,\ \forall y \in (b,c) \\ 0 \leq b < c \leq 1,\ F^* \in \hat{S}_1 \end{cases}$$

2.2.4.4 Example

Solve the duel–game with silencers from section 0.2 in the case
$P_1(x) = P_2(x) = 1 - x$.

2.3 An application—struggle for a market

Player P_2 controls two marketplaces. Player P_1 will oust P_2 from at least one of the two places. Each of the two players will spend 1 unit for a publicity campaign (advertising etc.). The value of the two places is characterized by the numbers $k_1 > 0$ resp. $k_2 > 0$ (measured for instance by the turnover).

P_1 captures that of the two marketplaces where he stakes more funds than P_2.

Let us denote by x and y the amount of funds of P_1 resp. P_2 that they spend for the first of the two places. Then the struggle can be modelled as a game over the unit square with the payoff–function

$$H(x,y) = \begin{cases} k_1(x-y) & , \quad x \geq y \\ k_2((1-x)-(1-y)) & , \quad x \leq y\ . \end{cases}$$

Solution:

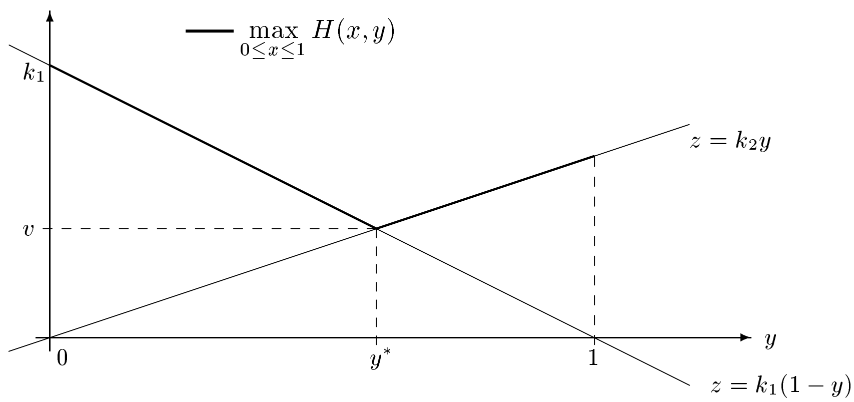
1. Function H is continuous. By theorem 2.8 we conclude that the game has a value v and (mixed) optimal strategies.
   For fixed x the function $H(x,\cdot)$ is convex. Therefore player P_2 has a pure optimal strategy (theorem 2.13).

2. From theorem 2.13 we know

$$v = \min_{0\le y\le 1} \max_{0\le x\le 1} H(x,y) \quad .$$

   We get

$$\begin{aligned}\max_{0\le x\le 1} H(x,y) &= \max\{\max_{x\ge y} k_1(x-y),\ \max_{x\le y} k_2(y-x)\}\\ &= \max\{\,k_1(1-y),\ k_2 y\,\} \quad .\end{aligned}$$



   Our figure shows, that the outer extremum in $v = \min_{0\le y\le 1} \max_{0\le x\le 1} H(x,y)$ is attained at $y^* = \frac{k_1}{k_1+k_2}$ and $v = \frac{k_1\cdot k_2}{k_1+k_2}$.
   I_{y^*} is an optimal strategy of P_2 (compare theorem 2.13).

3. What can we say about an optimal strategy F^* of P_1?
   We can not expect, that P_1 has a pure optimal strategy, because $H(\cdot,y)$ is not concave. But $\Psi(y) = H(\hat{x},y)$ is directionally differentiable. We can apply part (iii) from theorem 2.13. We are in the case $0 < y^* < 1$.
   $H(\hat{x},y^*) = v$ delivers

$$k_1\left(x - \frac{k_1}{k_1+k_2}\right) = \frac{k_1\cdot k_2}{k_1+k_2} \quad \text{if} \quad \hat{x} \ge y^*,$$

$$k_2\left(\frac{k_1}{k_1+k_2} - x\right) = \frac{k_1\cdot k_2}{k_1+k_2} \quad \text{if} \quad \hat{x} \le y^*.$$

   In the first case we get $x = x^{**} = 1$, in the second case $x = x^* = 0$.

$$\Psi(y) \stackrel{\text{def}}{=} H(x^*,y) = k_2 y \quad \Rightarrow \quad \Psi'(y^*;+1) = k_2$$

$$\hat{\Psi}(y) \stackrel{\text{def}}{=} H(x^{**},y) = k_1(1-y) \quad \Rightarrow \quad \hat{\Psi}'(y^*;+1) = -k_1$$

Finally we try to calculate F^* as

$$F^*(x) = \alpha \cdot I_{x^*}(x) + (1-\alpha) \cdot I_{x^{**}}(x),$$

where $\alpha\Psi(y^*;1) + (1-\alpha)\hat{\Psi}'(y^*;1) = 0$. The latter gives

$$\alpha \cdot k_2 + (1-\alpha)(-k_1) = 0,$$

from which we conclude, that $\alpha = \frac{k_1}{k_1+k_2}$.
$F^*(x) = \frac{k_1}{k_1+k_2} \cdot I_0(x) + \frac{k_2}{k_1+k_2} \cdot I_1(x)$ is an optimal strategy of player P_1.

4. Interpretation of our solution:
   Player P_2 spends an amount of $y^* = \frac{k_1}{k_1+k_2}$ for the first maket and an amount of $(1-y^*) = \frac{k_2}{k_1+k_2}$ for the second market, that is the greater part of his means is directed to the market with the greater value (consider the two cases $k_1 \leq k_2$ and $k_1 < k_2$).

   Player P_1 throws with probability $\frac{k_1}{k_1+k_2}$ an amount of zero into the first market (i.e. all means are spent to the second market). With probability $\frac{k_2}{k_1+k_2}$ he throws an amount of 1 (i.e. all means) into the first market (i.e. zero into the second market).

Chapter 3

Noncooperative n–person games

So far we considered antagonistic games, that is constant–sum games. Now we will allow the game not to be a constant–sum game. Such games fall into two classes: cooperative games and noncooperative games.

In the first class the rules of the game permit to form coalitions of players: The players may negotiate and groups of them may cooperate. Such games are subject of a separate course.

We consider here the second class. In such games the players act as individuals. The rules of such games do not allow to negotiate in order to reach agreements about behaviour in the game or redistribution of the individual payoffs.
We refer to section 2.1.1, where we stated the first notations about noncooperative games (normal form, Nash–equilibrium, strategically equivalent games, constant–sum games). We mostly restrict ourselves to two–person noncooperative games.

3.1 Three examples

3.1.1 Prisoner's dilemma

In section 0.3 we already introduced this finite two–person noncooperative game. In the normal form the payoff–functions H_1 and H_2 of player P_1 (Bonnie) and P_2 (Clyde) are given by the matrices

$$H_1 = \begin{pmatrix} -2 & -10 \\ -1 & -5 \end{pmatrix} \quad , \qquad H_2 = \begin{pmatrix} -2 & -1 \\ -10 & -5 \end{pmatrix} \quad .$$

We observe that $h_{ij}^1 + h_{ij}^2 \neq const.$

3.1.2 Battle of the sexes

On saturday evening the husband (player P_1) wishes to go to the football match and the wife (player P_2) to the ballet. Neither wishes to go alone to his or her own preferred choice. However, the rules of this non–cooperative game forbid any pre–play communication and they must make their choices independently. Let strategy 1 denote the choice of football match and strategy 2 that of ballet for both players. The satisfaction derived in each case may be estimated by the elements in the following payoff–matrices:

$$H_1 = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \quad , \qquad H_2 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \quad .$$

Again we have $h^1_{ij} + h^2_{ij} \neq const.$

3.1.3 Chicken (modern form of a duel)

Two american teenage males with cars meet at a lonely stretch of straight road. They position the cars a mile apart, facing each other, and drive toward each other at a high rate of speed. The cars straddle the center line of the road. If one of the drivers swerves before the other, then the swerver is called "chicken" and loses the respect of his peers. The nonswerver, on the other hand, gains prestige. If both swerve, neither is considered very brave but neither really loses face. If neither swerves, they both die. We assign numerical values to the various outcomes. Death is valued -10, being "chicken" is 0, not swerving when the other driver does is worth 5, swerving at the same time as the other is valued at 3.

$$H_1 = \begin{pmatrix} 3 & 0 \\ 5 & -10 \end{pmatrix} \quad , \qquad H_2 = \begin{pmatrix} 3 & 5 \\ 0 & -10 \end{pmatrix} \quad .$$

All of the examples are so–called bi–matrix games.

Definition 3.1. *Let $\Gamma = \langle P_1, P_2, S_1, S_2, h_1, H_2 \rangle$ be a noncooperative two–person game in normal form. If both sets S_1 and S_2 are finite sets (i.e. $card(S_1) < \infty$, $card(S_2) < \infty$) then we call Γ a* **bi–matrix game**.

Remark: Without loss of generality we may assume, that $S_1 = \{1, 2, \ldots, m\}$, $S_2 = \{1, 2, \ldots, n\}$. Denoting $H_1(i,j) =: a_{ij}$ and $H_2(i,j) =: b_{ij}$ we can introduce the payoff–functions H_1 and H_2 by fixing the two matrices

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \multicolumn{4}{c}{\dotfill} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ and } \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \multicolumn{4}{c}{\dotfill} \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} .$$

3.2 The Nash–solution concept

In antagonistic games a situation is accepted as a solution (optimal strategies) of the game if the situation is an equilibrium (cf. definition 2.3 and also definitions 1.2 and 1.4). In noncooperative games the same concept is applicable.

But it is not accepted in general as the only one. That's why we will consider other concepts, too. Each of the concepts has its advantages and drawbacks. This is the reason why we do not connect the words "optimal strategies" with any of the concepts. First we repeat the definition of an equilibrium (cf. definition 2.3). In connection with noncooperative games (i.e. nonconstant–sum games in general) we will add the word Nash to the word equilibrium.

Definition 3.2. *Let* $\Gamma = \langle P_1, P_2, \ldots, P_n, S_1, S_2, \ldots, S_n, H_1, H_2, \ldots, H_n \rangle$ *be a noncooperative n–person game in normal form.*
A situation $\sigma^* = [\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*] \in \prod_{i=1}^{n} S_i$ *is called a* **Nash–equilibrium** *in* Γ, *if it is acceptable for any of the n players, that is* $\forall i \in \{1, 2, \ldots, n\}$ *and any* $\sigma_i \in S_i$ *the inequality*

$$H_i(\sigma^*/\sigma_i) \leq H_i(\sigma^*) \text{ holds.}$$

Here σ^*/σ_i *denotes the situation* $[\sigma_1^*, \sigma_2^*, \ldots, \sigma_{i-1}^*, \sigma_i, \sigma_{i+1}^*, \ldots, \sigma_n^*]$ *(all players, except* P_i*, observe their strategy from situation* σ^**).*

In a bi–matrix game $[i_0, j_0]$ forms a Nash–equilibrium, if

$$(*) \qquad \begin{cases} a_{ij_0} \leq a_{i_0 j_0} & , \quad \forall i \in S_1 \\ b_{i_0 j} \leq b_{i_0 j_0} & , \quad \forall j \in S_2 \end{cases} .$$

In the prisoner's dilemma the situation $[i_0, j_0] = [2, 2]$ is the only Nash–equilibrium (both prisoners confess) and this situation may be realized by individual decisions.

In the "battle of the sexes" game we have two Nash–equilibria $[i_0, j_0] = [1, 1]$ and $[i_1, j_1] = [2, 2]$ (both go to football or both go to the ballet). In the "chicken" game both situations $[i_0, j_0] = [1, 2]$ and $[i_1, j_1] = [2, 1]$ form a Nash–equilibrium (only one of the drivers swerves).
But in both games it is difficult to imagine, how at least one of these equilibria will be realized by individual decisions (i.e. without collaboration):
For example, in the "battle of the sexes" game player P_1 will prefer situation $[1, 1]$, i.e. his individual choice will be $i_0 = 1$, and player P_2 will prefer situation $[2, 2]$, i.e. her individual choice will be $j_0 = 2$. But $[1, 2]$ is not a Nash–equilibrium.

In the chapters 1 and 2 we mentioned five advantages of the Nash–equilibrium if the game is an antagonistic one (and a Nash–equilibrium exists):

1. Stability: There is no reason why a player should defect (indivually) from the equilibrium,
2. Rectangular property,
3. Cooperation and agreements between the players can not improve the payoff for both players,
4. If both players deviate from the equilibrium then it is impossible that both will gain from this,
5. The pessimistic approach (P_1 plays the max min-strategy, P_2 plays the min max-stategy) leads to the equilibrium.

A view of our three examples shows that each of the five properties may be not true in nonantagonistic games. This circumstance led to a more stiffer term of equilibrium:

Definition 3.3. *Let* $\Gamma = \langle P_1, P_2, \ldots, P_n, S_1, S_2, \ldots, S_n, H_1, H_2, \ldots, H_n\rangle$ *be a noncooperative n–person game. Let* $\sigma^* = [\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*]$ *be a Nash–equilibrium in* Γ*. For* $K \subset \{1, 2, \ldots, n\}$ *we denote by* σ_K *an element* $\sigma_K = \{\sigma_i\}_{i\in K} \in \prod_{i\in K} S_i$ *and by* σ^*/σ_K *a situation* $\sigma^*/\sigma_K = [\sigma_1, \sigma_2, \ldots, \sigma_n] \in \prod_{i=1}^{n} S_i$*, where*

$$\sigma_i = \begin{cases} \sigma_i^* & , \quad \text{if } i \notin K \\ \sigma_i & , \quad \text{if } i \in K \end{cases} .$$

We call σ^* *a* **strong Nash–equilibrium** *in* Γ*, if* $\forall K \subset \{1, 2, \ldots, n\}$ *and* $\forall \sigma_K \in \prod_{i\in K} S_i$ *the inequalitiy*

$$(\circ) \qquad \sum_{i\in K} H_i(\sigma^*) \geq \sum_{i\in K} H_i(\sigma^*/\sigma_K)$$

holds.

Remarks:

1. If in Γ a strong equilibrium exists, then there is no sense to form any coalition of players: For the players P_i, $i \in K$, it makes sense to form a coalition, if there exists an $\sigma_K \in \prod_{i\in K} S_i$ such that

$$(+) \qquad \begin{cases} H_i(\sigma^*/\sigma_K) \geq H_i(\sigma^*), \quad \forall i \in K \\ \text{and at least one of the } card(K) \\ \text{inequalities is a strong one.} \end{cases}$$

   But, if σ^* satisfies $(\circ)$ (i.e. is a strong Nash–equilibrium), then $(+)$ contradicts $(\circ)$.

2. In the bi–matrix game

$$H_1 = \begin{pmatrix} 11 & 1 \\ 2 & 6 \end{pmatrix} , \qquad H_2 = \begin{pmatrix} 4 & 1 \\ 2 & 8 \end{pmatrix}$$

   the situations $[i_0, j_0] = [1, 1]$ and $[i_1, j_1] = [2, 2]$ both form a Nash–equilibrium. $[i_0, j_0]$ is a strong Nash–equilibrium. In the "battle of the sexes" game both Nash–equilibria are strong (but not really stronger). None of the Nash–equilibria in the "chickens" and "prisoners dilemma" games are strong.

3.3 The Pareto–solution concept

Definition 3.4. *Consider the noncooperative game*
$\Gamma = \langle P_1, P_2, \ldots, P_n, S_1, S_2, \ldots, S_n, H_1, H_2, \ldots, H_n\rangle$*. The solution* $\sigma^* \in \prod_{i=1}^{n} S_i$ *is*

said to be a **Pareto–solution** *if there is no other situation* $\sigma \in \prod_{i=1}^{n} S_i$ *such that*

$$\begin{cases} H_i(\sigma) \geq H_i(\sigma^*), \ \forall i = 1, 2, \ldots, n \\ \exists i_0 \in \{1, 2, \ldots, n\} \ : \ H_{i_0}(\sigma) > H_{i_0}(\sigma^*) \end{cases} .$$

In contrast to the Nash–equilibrium the Pareto–concept stands for a more collective behaviour: $\forall \sigma \in \prod_{i=1}^{n} S_i$ we have either $H_i(\sigma) = H_i(\sigma^*)$, $\forall i$, or $\exists i_1$ with $H_{i_1}(\sigma) < H_{i_1}(\sigma^*)$.

In a bi–matrix game $[i_0, j_0]$ is a Pareto–solution if the system

$$\begin{cases} a_{ij} \geq a_{i_0 j_0} \\ b_{ij} \geq b_{i_0 j_0} \\ \text{at least one of the two inequalities} \\ \ \text{is strong} \end{cases}$$

has no solution $[i, j] \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$.
In the prisoner's dilemma only the situation $[i, j] = [2, 2]$ is not a Pareto–solution. In the "battle of the sexes" game $[i_0, j_0] = [1, 1]$ and $[i_1, j_1] = [2, 2]$ are both Pareto–solutions. Finally in the "chickens" game only $[i, j] = [2, 2]$ is not a Pareto–solution.

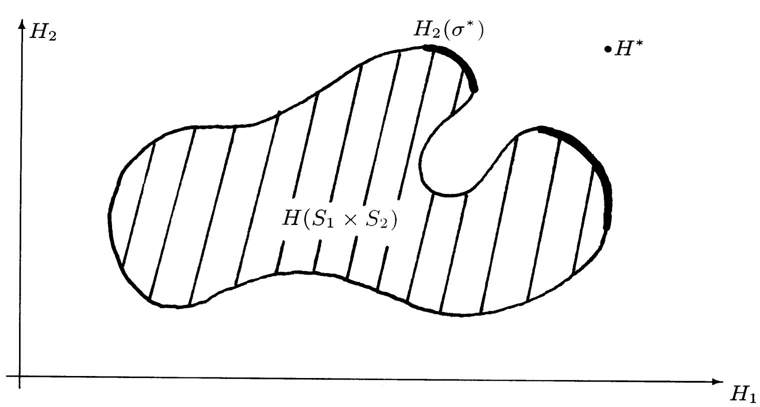
A disadvantage of the Pareto–concept is that usually the set of Pareto–solutions is very rich.

Proposition 3.1. *A strong Nash–equilibrium in a n–person noncooperative game is always a Pareto–solution.*

Suppose for a moment, that $n = 2$ and that S_1 and S_2 are continuous (i.e. infinite sets). We set

$$H(x, y) = [\, H_1(x, y),\, H_2(x, y)].$$

In the payoff–space $[H_1, H_2]$ the whole image of $S_1 \times S_2$ by the mapping $H : S_1 \times S_2 \to \mathbb{R}^2$ will be a set like in our figure.



The images of Pareto–solutions are shown by thick lines. But, generally it is not imaginable how such a Pareto–solution should be realized by individual decisions of the players.
The figure shows that, generally, the selection of a Pareto–solution as the solution of a game is not a very precise mechanism.

Suppose, for example, that there exists a tupel $\sigma^* = [\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*] \in \prod_{i=1}^{n} S_i$ which maximizes the payoff–function H_{i_0} of the i_0-th player, i.e.

$$H_{i_0}(\sigma^*) = \max\left\{ H_{i_0}(\sigma) \;:\; \sigma \in \prod_{i=1}^{n} S_i \right\}.$$

In most cases σ^* will be a Pareto–solution. But the other players will not be very pleased with situation σ^* (cf. σ^* in our figure in the case if $i_0 = 2$, cf. also our examples from section 3.1).
σ^* is the best candidate for a solution scheme if it is an optimal solution of the $(n-1)$ remaining problems

$$\max\left\{ H_i(\sigma) \;:\; \sigma \in \prod_{i=1}^{n} S_i \right\}, \; i \neq i_0,$$

too, what only happens in exceptional cases.

Proposition 3.2. *Let us consider the case that* $H(S_1, S_2, \ldots, S_n) = \left\{ [H_1(x), H_2(x), \ldots, H_n(x)] \;:\; x \in \prod_{i=1}^{n} S_i \right\}$ *is a closed convex set. Suppose that the so–called virtual maximum of the game* $H^* = [H_1^*, H_2^*, \ldots, H_n^*]$ *exists (i.e. its components are finite), where*

$$H_i^* = \sup\left\{ H_i(\sigma) \;:\; \sigma \in \prod_{i=1}^{n} S_i \right\}, \; i = 1, 2, \ldots, n.$$

Then the situation $\bar{\sigma} \in \prod_{i=1}^{n} S_i$ *with*

$$\|H(\bar{\sigma}) - H^*\| = \min\left\{ \|H(\sigma) - H^*\| \;:\; \sigma \in \prod_{i=1}^{n} S_i \right\}$$

is a Pareto–solution of the game.

But, even if $\bar{\sigma}$ is unique (under the conditions of the proposition the vector $H(\bar{\sigma})$ is unique) it can be realized only by the way of collaboration.
Pareto–solutions arise in vector–optimization, too. There often a weight $\lambda_i > 0$ is attributed to each player P_i and then it is proved that an optimal solution of the weighted function–problem

$$\max\left\{ \sum_{i=1}^{n} \lambda_i H_i(x) \;:\; x \in \prod_{i=1}^{n} S_i \right\} \qquad (\triangle)$$

is a Pareto–solution of the vector–optimization problem.
The same is true in game theory. But, accepting the weights and accepting problem $(\triangle)$ presupposes some sort of collaboration among the players, which contradicts with the axioms of noncooperative game theory.

3.4 The conservative concept

For simplicity-reasons let us consider again the case if $n = 2$. Let be given the noncooperative game $\Gamma = \langle P_1, P_2, S_1, S_2, H_1, H_2 \rangle$. We put

$$v_1 := \sup_{x \in S_1} \inf_{y \in S_2} H_1(x, y)$$
$$v_2 := \sup_{y \in S_2} \inf_{x \in S_1} H_2(x, y).$$

If we suppose the existence of an x^* and an y^* that realize the outer extrema, i.e.

$$(\Box) \qquad \inf_{y \in S_2} H_1(x^*, y) = v_1, \qquad \inf_{x \in S_1} H_2(x, y^*) = v_2,$$

then the situation $[x^*, y^*]$ has the property

$$\begin{cases} H_1(x^*, y^*) \geq v_1 \\ H_2(x^*, y^*) \geq v_2 \end{cases}.$$

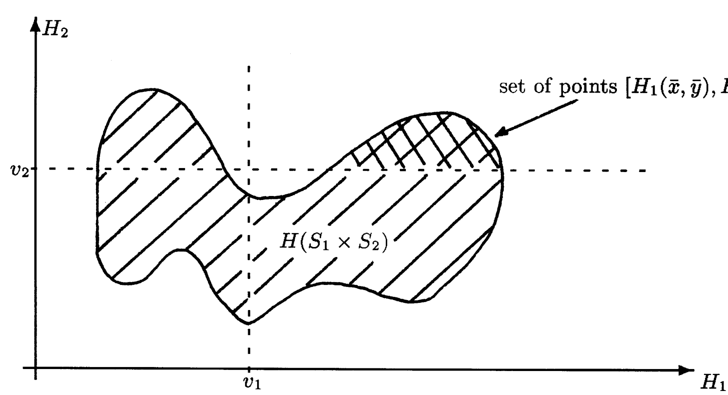
The concept which accepts as a solution of Γ only a situation $[\bar{x}, \bar{y}] \in S_1 \times S_2$ with

$$(\Box\Box) \qquad \begin{cases} H_1(\bar{x}, \bar{y}) \geq v_1 \\ H_2(\bar{x}, \bar{y}) \geq v_2 \end{cases}$$

is called the **conservative concept**.
But, only the choice $\bar{x} := x^*$, $\bar{y} := y^*$ may be realized in the sence of a non-cooperative game (if x^* and y^* are unique solutions of $(\Box)$). All other choices satisfying $(\Box\Box)$ need some cooperation between P_1 and P_2.

In the "prisoner's dilemma" game the conservative choice is $[i_0, j_0] = [2, 2]$, and in the "battle of sexes" game we get that all possible solutions $[1, 1]$, $[1, 2]$, $[2, 1]$, $[2, 2]$ are a conservative choice. Finally, in the "chicken" game $[i_0, j_0] = [1, 1]$ is the conservative choice.

As we can see from the illustration of the game in the $H_1 - H_2$–space (cf. section 3.3), the conservative choice $[x^*, y^*]$, generally, will not be a Pareto–solution (i.e. there exist, in general, situations which are better for both players).



3.5 The Stackelberg-concept

We shall now consider behaviour of a "leader–follower" type in a noncooperative two–person game $\Gamma = \langle P_1, P_2, S_1, S_2, H_1, H_2 \rangle$. Denote by Z^1 and Z^2 the sets of best response for players P_1 and P_2, respectively, where

$$\begin{aligned} Z^1 &:= \{[\sigma_1, \sigma_2] \in S_1 \times S_2 \,:\, H_1(\sigma_1, \sigma_2) = \max_{y_1 \in S_1} H_1(y_1, \sigma_2)\} \\ Z^2 &:= \{[\sigma_1, \sigma_2] \in S_1 \times S_2 \,:\, H_2(\sigma_1, \sigma_2) = \max_{y_2 \in S_2} H_2(\sigma_1, y_2)\} \end{aligned}$$

(suprema are supposed to be reached).

Definition 3.5. *We call the situation $\sigma^* = [\sigma_1^*, \sigma_2^*] \in S_1 \times S_2$ the* **Stackelberg $\mathbf{P_1}$–equilibrium** *in the game Γ if $\sigma^* \in Z^2$ and*

$$H_1(\sigma^*) = \max_{\sigma \in Z^2} H_1(\sigma) \qquad (\triangle)$$

holds.
Analogously we define a P_2–equilibrium in Γ ($\sigma^* \in Z^1$, $H_2(\sigma^*) = \max_{\sigma \in Z^1} H_2(\sigma)$).

Remarks:

1. The notion P_1–equilibrium may be interpreted as follows.
   Player P_1 (the leader) knows the payoff functions of both players, and hence he learns player P_2's (follower) set of best responses Z_2 to any strategy y_1 of player P_1. Having this information he then maximizes his payoff by selecting strategy y_1 from condition ($\triangle$).

2. Thus, $H_1(\sigma^*)$ is a payoff to the player P_1 acting as a "leader" in the game Γ.

3. σ^* is called Stackelberg–equilibrium after the economist Heinrich Freiherr von Stackelberg who described this behaviour in 1933 in a review of price theory.

4. Example: Consider the finite game with

   $$H_1 = \begin{pmatrix} 11 & 1 \\ 2 & 6 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 4 & 1 \\ 2 & 8 \end{pmatrix}.$$

   Then we have

   $$\begin{aligned} Z^1 &= \{\,(1,1)\,,\,(2,2)\,\}, \\ Z^2 &= \{\,(1,1)\,,\,(2,2)\,\}. \end{aligned}$$

   The P_1–equilibrium is $(1,1)$, whereas $(2,2)$ is the P_2–equilibrium. Note, that $Z^1 \cap Z^2$ forms the set of Nash–equilibria in our example.

Corollary 3.3.
Consider the noncooperative 2–person game $\Gamma = \langle P_1, P_2, S_1, S_2, H_1, H_2\rangle$. Denote by Z the set of Nash–equilibria in Γ. Then

$$Z = Z^1 \cap Z^2 \qquad\qquad \textit{holds.}$$

Definition 3.6 (Moulin 1981). *Let σ^* be a P_1–equilibrium and σ^{**} a P_2–equilibrium in the game $\Gamma = \langle P_1, P_2, S_1, S_2, H_1, H_2\rangle$. We say that the game Γ involves* **competition for leadership** *if there does not exist a situation $\bar\sigma \in S_1 \times S_2$ such that*

$$(\nabla) \quad \begin{cases} H_1(\sigma^*) \le H_1(\bar\sigma) \\ H_2(\sigma^{**}) \le H_2(\bar\sigma) \end{cases}$$

Remarks:

1. If $\sigma^* = \sigma^{**}$ happens, then (∇) is true with $\bar\sigma = \sigma^* = \sigma^{**}$ and therefore there is no competition for leadership.
2. Competition for leadership is involved, for instance, in the game with

$$H_1 = \begin{pmatrix} 11 & 1 \\ 2 & 6 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 4 & 1 \\ 2 & 8 \end{pmatrix}.$$

Theorem 3.4 (Moulin 1981). *Consider the noncooperative two–person game Γ. If Γ has at least two situations σ^* and σ^{**} with the properties*

$$\begin{cases} \sigma^*,\ \sigma^{**} \textit{ are both Nash–equilibria in } \Gamma, \\ \sigma^*,\ \sigma^{**} \textit{ are both a Pareto solution of } \Gamma, \\ [H_1(\sigma^*), H_2(\sigma^*)] \ne [H_1(\sigma^{**}), H_2(\sigma^{**})] \end{cases}$$

then the game involves competition for leadership.

3.6 Existence of a Nash–equilibrium

Let be given a noncooperative n–person game
$\Gamma = \langle P_1, P_2, \ldots, P_n, S_1, S_2, \ldots, S_n, H_1, H_2, \ldots, H_n\rangle$,
where

$$S_i \subset \mathbb{R}^{n_i} \quad , \quad i = 1, 2, \ldots, n \ ,$$
$$H_i : \prod_{i=1}^{n} S_i \to \mathbb{R} \quad , \quad i = 1, 2, \ldots, n \ .$$

We denote by F_i a probability measure given on Borel σ-algebras on the set S_i and by $\hat S_i$ the set of all these probability measures, $i = 1, 2, \ldots, n$.

Definition 3.7. *We call the noncooperative n–person game*

$$\hat\Gamma = \langle P_1, P_2, \ldots, P_n, \hat S_1, \hat S_2, \ldots, \hat S_n, \hat H_1, \hat H_2, \ldots, \hat H_n\rangle$$

the **mixed extension** *of the game* Γ *if for* $i = 1, 2, \ldots, n$

$$\begin{cases} \hat{H}_i : \prod_{i=1}^{n} \hat{S}_i \to \mathbb{R} \quad \textit{and} \\ \hat{H}_i(F_1, F_2, \ldots, F_n) = \int\limits_{S_n} \int\limits_{S_{n-1}} \ldots \int\limits_{S_2} \int\limits_{S_1} H_i(\sigma^1, \sigma^2, \ldots, \sigma^n) dF_1(\sigma^1) \\ \qquad\qquad\qquad\qquad\qquad\qquad dF_2(\sigma^2) \ldots dF_n(\sigma^n). \end{cases}$$

Remarks:

1. In $\hat{H}_i(F_1, F_2, \ldots, F_n)$ the integrals are taken to be Lebesque–Stieltjes integrals.
2. If F_1, respectively F_2,..., respectively F_n are the mixed strategies of player P_1, resp. player P_2,..., resp. player P_n in game $\hat{\Gamma}$, then the payoff $\hat{H}_i(F_1, F_2, \ldots, F_n)$ is the mathematical expectation of payoff of player P_i in the situation $[F_1, F_2, \ldots, F_n]$ in game $\hat{\Gamma}$.
3. The definition of a Nash–equilibrium from section 2.1.1 is applicable not only to game Γ but to game $\hat{\Gamma}$ too. In this connection we call a $\sigma^i \in S_i$ a pure strategy of player P_1 and an $F_i \in \hat{S}_i$ a mixed strategy of the same player P_i.
4. Consider the special case that all S_i are finite sets, for definiteness let be

$$S_i = \{x_i^1, x_i^2, \ldots, x_i^{m_i}\},\ i = 1, 2, \ldots, n\,.$$

   Then

$$\hat{S}_i = \left\{ [p_i^1, p_i^2, \ldots, p_i^{m_i}] \,:\, p_i^j \geq 0,\ \forall j,\ \sum_{j=1}^{m_i} p_i^j = 1 \right\}$$

   and

$$\hat{H}_i(p_1, p_2, \ldots, p_n) = \sum_{j_n=1}^{m_n} \ldots \sum_{j_2=1}^{m_2} \sum_{j_1=1}^{m_1} H_i(x_1^{j_1}, x_2^{j_2}, \ldots, x_n^{j_n}) \cdot p_1^{j_1} p_2^{j_2} \ldots p_n^{j_n}\,,$$

   where p_i denotes the vector $p_i = [p_i^1, p_i^2, \ldots, p_i^{m_i}],\ i = 1, 2, \ldots, n.$

5. In the case of a bi–matrix game $\Gamma = \langle P_1, P_2, S_1, S_2, H_1, H_2 \rangle$
   with $S_1 = \{1, 2, \ldots, m\}, \quad S_2 = \{1, 2, \ldots, n\}$
   and $H_1 = \{a_{ij}\}_{i=1,j=1}^{m,n}, \quad H_2 = \{b_{ij}\}_{i=1,j=1}^{m,n}$
   we get as the mixed extension the game $\hat{\Gamma} = \langle P_1, P_2, \hat{S}_1, \hat{S}_2, \hat{H}_1, \hat{H}_2 \rangle$ with

$$\begin{aligned} \hat{S}_1 &= \{x \in \mathbb{R}^m \,:\, x \geq 0,\ \langle e, x \rangle = 1\}, \\ \hat{S}_2 &= \{y \in \mathbb{R}^n \,:\, y \geq 0,\ \langle e, y \rangle = 1\}, \end{aligned}$$

$$\hat{H}_1(x, y) = \langle x, H_1 y \rangle, \qquad \hat{H}_2(x, y) = \langle x, H_2 y \rangle$$

   (e denotes a vector of 1's).

Theorem 3.5. *(cf. lemma 1.7)*
Let $\Gamma = \langle P_1, \dots, P_n, S_1, \dots, S_n, H_1, \dots H_n \rangle$ *be a finite noncooperative n–person game. We denote by* $\hat{\Gamma}$ *the mixed extension of* Γ.
Then the following is true:

$$F^* = [F_1^*, F_2^*, \dots, F_n^*] \in \prod_{i=1}^{n} \hat{S}_i$$

is a Nash–equilibrium in $\hat{\Gamma}$ *if and only if*

$$\hat{H}_i(F_1^*, \dots, F_{i-1}^*, I_\sigma, F_{i+1}^*, \dots, F_n^*) \le \hat{H}_i(F^*), \quad \forall \sigma \in S_i, \ \forall i = 1, 2, \dots, n.$$

(The definition of $I_\sigma(x_i)$ *was given in section 2.2.1)*

Theorem 3.6 (Nash 1950).
In the mixed extension $\hat{\Gamma}$ *of a finite noncooperative n–person game* $\Gamma = \langle P_1, \dots, P_n, S_1, \dots, S_n, H_1, \dots H_n \rangle$ *a Nash–equilibrium always exists.*

This theorem generalizes the main theorem of matrix games (theorem 1.8) to finite noncooperative n–person games.

Theorem 3.7. *(generalization of theorem 2.8)*
Let $\Gamma = \langle P_1, \dots, P_n, S_1, \dots, S_n, H_1, \dots H_n \rangle$ *be a noncooperative n–person game, where*

> S_i *is a closed, bounded set in the finite dimensional space* $\mathbb{R}^{m_i}$, $i = 1, 2, \dots, n$

and where H_i *is a continous function on* $\prod_{i=1}^{n} S_i$.

Then in the mixed extension $\hat{\Gamma}$ *of* Γ *a Nash–equilibrium exists.*

Theorem 3.8. *(generalization of corollary 2.12)*
Let $\Gamma = \langle P_1, \dots, P_n, S_1, \dots, S_n, H_1, \dots H_n \rangle$ *be a noncooperative n–person game, where*

> S_i *is a closed, bounded, convex set in a finite dimensional space* $\mathbb{R}^{m_i}$, $i = 1, 2, \dots, n$

and where

> $H_i(\sigma^1, \dots, \sigma^{i-1}, \sigma^i, \sigma^{i+1}, \dots, \sigma^n)$ *is a concave function of* σ^i *over* S_i *for every fixed* $(n-1)$*-tupel* $[\sigma^1, \dots, \sigma^{i-1}, \sigma^{i+1}, \dots, \sigma^n] \in \prod_{\substack{k=1 \\ k \neq i}}^{n} S_k$, $i = 1, 2, \dots, n$.

Then in Γ *a Nash–equilibrium exists (i.e. in pure strategies).*

Definition 3.8. *Let* $\hat{\Gamma}$ *be the mixed extension of the noncooperative n–person game* $\Gamma = \langle P_1, \dots, P_n, S_1, \dots, S_n, H_1, \dots H_n \rangle$. *The pure strategy* $\sigma \in S_i$ *we call a* **spectral point** *of the mixed strategy* $F_i \in \hat{S}_i$ *of player* P_i, *if for any neightbourhood* $U(\sigma)$ *of* σ *we have that*

$$\int_{U(\sigma) \cap S_i} dF_i(x) > 0 \, .$$

Theorem 3.9. *(generalization of theorem 2.10)*
Let $F^* = [F_1^*, F_2^*, \ldots, F_n^*] \in \prod_{i=1}^{n} \hat{S}_i$ *be a Nash–equilibrium in the mixed extension* $\hat{\Gamma}$ *of the noncooperative n–person game* $\Gamma = \langle P_1, \ldots, P_n, S_1, \ldots, S_n, H_1, \ldots H_n \rangle$.
Suppose that

$$H_i(\sigma^1, \ldots, \sigma^{i-1}, \sigma^i, \sigma^{i+1}, \ldots, \sigma^n)$$

is a continuous function of σ^i *on* S_i *for every fixed* $(n-1)$*-tupel* $[\sigma^1, \ldots, \sigma^{i-1}, \sigma^{i+1}, \ldots, \sigma^n] \in \prod_{\substack{k=1 \\ k \neq i}}^{n} S_k$.
Then the following is true:
If σ^i *is a spectral point of* F_i^**, then*

$$(*) \qquad \hat{H}_i(F_1^*, \ldots, F_{i-1}^*, I_\sigma, F_{i+1}^*, \ldots, F_n^*) = \hat{H}_i(F^*).$$

Remarks:

1. In contrast to the similar result for antagonistic games (cf. theorem 2.10) equality $(*)$ holds for noncooperative games, generally, only in connection with the Nash–equilibrium F^* in which F_i^* occurs.

2. If there is some information about spectral points we may use $(*)$ in order to calculate a Nash–equilibrium. Especially this is useful in bi–matrix games.

3. In the bi–matrix game Γ with the matrices

$$H_1 = \begin{pmatrix} 11 & 1 \\ 2 & 6 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 4 & 1 \\ 2 & 8 \end{pmatrix}$$

   we earlier found the Nash–equilibria (in pure strategies) $[i_0, j_0] = [1, 1]$ and $[i_1, j_1] = [2, 2]$.
   With

$$x^* = [1, 0] \quad , \quad y^* = [1, 0] \quad \text{and}$$

$$x^{**} = [0, 1] \quad , \quad y^{**} = [0, 1]$$

   as mixed strategies of P_1, respectively P_2 we have that $[x^*, y^*]$ and $[x^{**}, y^{**}]$ are both Nash–equilibria in the mixed extension $\hat{\Gamma}$ of Γ.
   But this are not the only Nash–equilibria in $\hat{\Gamma}$. It is not difficult to prove that $[x^{***}, y^{***}]$ with $x^{***} = [\frac{2}{3}, \frac{1}{3}]$, $y^{***} = [\frac{5}{14}, \frac{9}{14}]$ is a Nash–equilibrium in $\hat{\Gamma}$ too and no other equilibria exist in $\hat{\Gamma}$.
   All three equilibria have different value (in contrast to equilibria in antagonistic games):

	$[x^*, y^*]$	$[x^{**}, y^{**}]$	$[x^{***}, y^{***}]$
value for P_1	11	6	32/7
value for P_2	4	8	10/3

4. In the "chickens"–game (cf. section 3.1.3) an analogous situation happens:

$$[i_0, j_0] = [1, 2] \text{ and } [i_1, j_1] = [2, 1]$$

are Nash–equilibria in pure strategies and $[x^{***}, y^{***}] = [5/6\,, 1/6]$ is a Nash–equilibrium in mixed strategies.

	$[x^*, y^*]$	$[x^{**}, y^{**}]$	$[x^{***}, y^{***}]$
value for P_1	0	5	5/2
value for P_2	5	0	5/2

5. In the actual section 3.6 we discussed the Nash–equilibrium in the mixed extension of an n–person game.
The Pareto–solution (cf. section 3.3) and the Stackelberg–solution (cf. section 3.4) may also be discussed for the mixed extension of an n–person game.

3.7 Calculation of Nash–equilibria in bi–matrix games via nonlinear programming

In this section we consider the mixed extension $\hat{\Gamma}$ of the bi–matrix game Γ with the matrices

$$A = \{a_{ij}\}_{i=1,j=1}^{m,n} \quad \text{and} \quad B = \{b_{ij}\}_{i=1,j=1}^{m,n}.$$

We have $\hat{\Gamma} = \langle P_1, P_2, \hat{S}_1, \hat{S}_2, \hat{H}_1, \hat{H}_2 \rangle$, where

$$\begin{cases} \hat{S}_1 = \{x \in \mathbb{R}^m \,:\, x \geq 0,\ \langle x, e\rangle = 1\}, \\ \hat{S}_2 = \{y \in \mathbb{R}^n \,:\, y \geq 0,\ \langle y, e\rangle = 1\}, \\ H_1(x,y) = \langle x, Ay\rangle, \\ H_2(x,y) = \langle x, By\rangle\ . \end{cases}$$

Here e denotes a vector $e = [1, 1, \ldots, 1]$.

$[x^*, y^*] \in \hat{S}_1 \times \hat{S}_2$ is a Nash–equilibrium in $\hat{\Gamma}$ if and only if

$$(\circ) \quad \begin{cases} \langle x^*, Ay^*\rangle \geq \langle x, Ay^*\rangle \quad , \quad \forall x \in \hat{S}_1 \\ \langle x^*, By^*\rangle \geq \langle x^*, Ay\rangle \quad , \quad \forall y \in \hat{S}_2 \end{cases} .$$

Lemma 3.10. *$[x^*, y^*] \in \hat{S}_1 \times \hat{S}_2$ is a Nash–equilibrium in $\hat{\Gamma}$ if and only if there exist two real numbers p and q such that*

$$(\Box) \quad Ay^* \leq p \cdot e \quad , \quad B^\top x^* \leq q \cdot e \quad , \quad \langle\, x^*, (A+B)y^*\rangle = p + q\ .$$

Consider the nonlinear (and, generally, nonconvex) mathematical programming problem

$$(BLOA) \quad \begin{cases} \langle\, x, Ay\rangle + \langle\, x, By\rangle - p - q \to \max\limits_{x,y,p,q} \\ Ay \leq p \cdot e \quad , \quad B^\top x \leq q \cdot e \\ y \geq 0 \qquad\qquad x \geq 0 \\ \langle\, x, e\rangle = 1 \qquad \langle\, y, e\rangle = 1 \end{cases}$$

Theorem 3.11. *$[x^*, y^*] \in \hat{S}_1 \times \hat{S}_2$ is a Nash–equilibrium in $\hat{\Gamma}$ if and only if the problem $(BLOA)$ has a global–optimal solution $[x, y, p, q]$ with $x = x^*,\ y = y^*$.*

We remark that in the case $B = -A$ (i.e. Γ is an antagonistic game) problem $(BLOA)$ decomposes into the two linear programming problems

$$\left\{ \begin{array}{c} p \to \min\limits_{y,p} \\ Ay \leq p \cdot e \\ y \geq 0 \\ \langle y, e \rangle = 1 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{c} q \to \min\limits_{q,x} \\ -A^\top x \leq q \cdot e \\ x \geq 0 \\ \langle x, e \rangle = 1 \end{array} \right.$$

which we know from the theory of matrix games.

Chapter 4

Games in extensive form (positional games)

4.1 From games in normal form to games in extensive form

The preceding chapters dealt with games in normal form. In such a game the participants have to take a decision only once. A dynamic (i.e. continued during a period of time) process can be reduced to a normal form by formal introduction of the notion of a pure strategy. In the few cases, when the power of a strategy space is not great and the possibility exists of numerical solutions, such an approach seems to be allowable.

However, in the majority of the problems the passage to normal form, i.e. the reduction of the problem to a single choice of pure strategies as elements of large dimension spaces, does not lead to effective ways of finding solutions.

We need an other description of such dynamic games that develop in discrete stages or continuously. Mathematical dynamic models of conflict are investigated in the theory of positional games.

We consider only the simplest class of positional games, the class of finite stage games with perfect information. To define such a game we need a rudimentary knownledge of graph theory.

4.2 Set–valued maps and graphs

Definition 4.1. *Let X be a finite set. The* **set–valued map** *F of the set X into X is the rule which sets up a correspondence between every element $x \in X$ and a subset $F_x \subset X$.*

Remarks:

1. $F_x = \emptyset$ is not ruled out.
2. F_x is the image of the element x by the map F.
3. Let $A \subset X$. By the image of set A we will mean the set
$$FA := \bigcup_{x \in A} F_x \ .$$
4. By definition, let $F(\emptyset) = \emptyset$.
5. It can be seen that if $A_i \subset X,\ i = 1, 2, \ldots, n$, then
$$F\left(\bigcup_{i=1}^{n} A_i\right) = \bigcup_{i=1}^{n} FA_i \ , \quad F\left(\bigcap_{i=1}^{n} A_i\right) = \bigcap_{i=1}^{n} FA_i \ .$$
6. The power of a map we introduce in a natural way:
$$F_x^2 = F(F_x) \ , \ F_x^3 = F(F_x^2) \ , \ddots , F_x^k = F(F_x^{k-1}) \ .$$
7. The set–valued map $\hat{F}$ of the set X into X is called the transfinite closure of the map F, if
$$\hat{F}_x = \{x\} \cup F_x \cup F_x^2 \cup \ldots \cup F_x^k \cup \ldots .$$
8. The map F^{-1} that is inverse to the map F is defined as
$$F^{-1}y := \{\, x \ : \ y \in F_x \,\}.$$
9. $\left(F^{-1}\right)^2 y := F^{-1}\left(F^{-1}y\right) , \left(F^{-1}\right)^3 y := F^{-1}\left((F^{-1})^2 y\right) \ , \ldots$
$\left(F^{-1}\right)^k y := F^{-1}\left((F^{-1})^{k-1} y\right) \ .$

Example: Chess as a set–valued map
Every position on a chess–board is defined by the number and composition of chess pieces for each player as well as by the arrangement of chess pieces at a given moment and the indication as to whose move it is.

Suppose X is the set of positions, F_x for an $x \in X$ is the set of those positions which can be realized immediately after the position x has been realized. If in the position x the number of black or white pieces is zero, than $F_x = \emptyset$. Now F_x^k is the set of positions which can be obtained from x into k moves, $\hat{F}_x$ is the set of all positions which can be obtained from x, $F^{-1}(A)$ (with $A \subset X$) is the set of all positions from which it is possible to make, in one move, the transition to positions of set A.

Depicting positions by dots and connecting by an arrow two positions x and y, $y \in F_x$, it is possible to construct the graph of a game emanating from the original position.

For chess however, it is impossible to draw such a graph in reality. But, for many multistage games (like chess, draughts, go,...) the use of set–valued maps and graphs is an appropriate form of description.

Definition 4.2. *The pair (X, F) is called a* **graph** *if X is a finite set and F is a (set–valued) map of X into X.*

Remarks:

1. We denote a graph by the capital letter G, for instance $G = (X, F)$.
2. In what follows, the elements of the set X are represented by points on a plane, and the pairs of points x and y, for which $y \in F_x$ are connected by the solid line with arrow pointing from x to y.
3. Every element of the set X is called a **node** of the graph, and the pair of elements (x, y), where $y \in F_x$ is called an **arc** of the graph.
4. Two arcs p and q are called **incident** if they are distinct and have a boundary point in common.
5. The set P of arcs in the graph $G = (X, F)$ determines the map F, and vice versa the map F determines the set P. Therefore, the graph G can be represented as
   $G = (X, F)$ or $G = (X, P)$.

Definition 4.3. *A* **path** *in the graph $G = (X, F)$ is called a sequence of arcs $p = (p_1, p_2, \ldots, p_k, \ldots)$ such that the end of the preceding arc coincides with the origin of the next one.*
The **length of the path** *$p = (p_1, \ldots, p_k)$ is the number k of arcs in the sequence.*

Definition 4.4. *We say that the graph $G = (X, F)$ is a* **tree***, if*

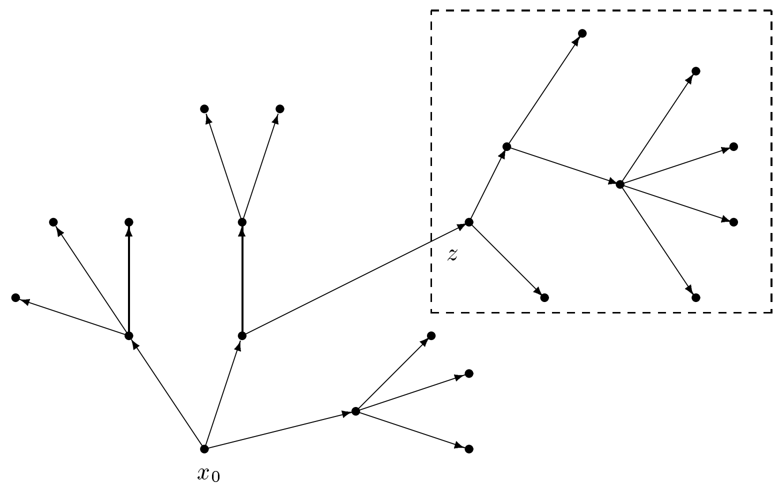
- *there exists a unique node $x_0 \in X$ with $\hat{F}_{x_0} = X$,*
- *$card(X) \geq 2$*
- *$card(X) < \infty$*
- *$\forall x \in X \backslash \{x_0\}$ there exists a unique path in G connecting x_0 with x.*

The node x_0 is called the **root of the tree***.*

Definition 4.5. *Let $z \in X$ and let the graph $G = (X, F)$ be a tree. The* **subgraph** *G_z is called the graph of the form (X_z, F_z), where $X_z := \hat{F}_z$ and $F_z x = F_x \cap X_z$.*

Remarks:

1. If G is a tree and $x \in X_z$, then the set $F_z x$ $(= F_x \cap X_x)$ and F_x coincide, i.e. the map F_z is a restriction of the map F to the set X_z. Therefore, for the subgraphs of the tree we use the notation $G_z = (X_z, F)$.
2. The following graph is a tree and the dotted line encircles the subgraph starting in the node z.



4.3 Multistage games with complete information

Definition 4.6. *We say that an* **n–person positional game** *is given if the following items are given:*

- *a tree* $G = (X, F)$,
- *a natural number* n *and* n *players* $P_1, P_2, \ldots, P_n$,
- *a partition of the node set* X *into* $n+1$ *sets* $X_1, X_2, \ldots, X_n, X_{n+1}$ *with* $\bigcup_{i=1}^{n+1} X_i = X,\ X_k \cap X_l = \emptyset$ *if* $k \neq l$,
  and $F_x = \emptyset$ *for* $x \in X_{n+1}$,
- n *real functions* $H_i(x)$ *with* $H_i : X_{n+1} \to \mathbb{R},\ i = 1, 2, \ldots, n$.

Remarks:

1. X_{n+1} is called the **set of final positions**
2. X_i is called the **priority set** for the i-th player P_i
3. H_i is called a payoff to the i-th player
4. The game proceeds as follows: Let the root x_0 of the tree belong to X_{i_1}. Then the player P_{i_1} "makes a move" and chooses the next node (position) $x_1 \in F_{x_0}$. If $x_1 \in X_{i_2}$, then player P_{i_2} "makes a move" and chooses the next node (position) $x_2 \in F_{x_1}$ and so on. Thus, if the node $x_{k-1} \in X_{i_k}$ is realized at the k-th step, then in this node player P_{i_k} "makes a move"

and selects the next node (position) from the set $F_{x_{k-1}}$.
The game terminates as soon as the terminal node $x_e \in X_{n+1}$ is reached. In the position x_e each of the players P_i receives a payoff $H_i(x_e),\ i = 1, 2, \ldots, n$.

5. Such a step–by–step selection implies a unique realization of some sequence $x_0, \ldots, x_k, \ldots, x_e$ determining a path in the graph G which emanates from the root position and reaches one of the final positions of the game. In what follows, such a path is called a **play of the game**.

   Because of the tree–like structure of the graph G, each play uniquely determines the final position x_e to be reached and, conversely, the final position x_e uniquely determines the play.

6. We assume that player P_i making his choice in position $x \in X_i$ knows his position and hence (G is a tree) can restore all previous positions. In this case, the players are said to have **complete information**.

Definition 4.7.
The single valued map u_i, which sets up a correspondence between each node $x \in X_i$ and some node $y \in F_x$ is called a **strategy** *for player P_i.*

Remarks:

1. A strategy u_i determines the "move" of the i-th player in each position $x \in X_i$.

2. The set of all possible strategies of P_i is denoted by U_i.

3. The ordered set $u = [u_1, u_2, \ldots, u_n]$, where $u_i \in U_i$ is called a **situation** in the game.

4. The cartesian product
$$U = \prod_{i=1}^{n} U_i$$
   is called the **set of situations**.

5. Each situation $u = [u_1, u_2, \ldots, u_n]$ uniquely determines a play in the game, and hence payoffs to the players.
   Suppose that the situation $u = [u_1, u_2, \ldots, u_n]$ corresponds to the play $x_0, x_1, \ldots, x_e$. Then we may introduce the notion of the payoff–function K_i for player P_i by equanting its value in each situation u to the value of the payoff H_i in the final position, that is
$$K_i(u_1, u_2, \ldots, u_n) := H_i(x_e), \quad i = 1, 2, \ldots, n\ .$$

6. Thus we obtain a game in normal form
$$\Gamma = \langle P_1, P_2, \ldots, P_n, U_1, U_2, \ldots, U_n, K_1, K_2, \ldots, K_n \rangle\ ,$$
   where $K_i : \prod_{i=1}^{n} U_i \to \mathbb{R},\ i = 1, 2, \ldots, n$.

4.4 Absolute equilibrium

Definition 4.8. *Let $z \in X$. Consider the subgraph $G_z = (X_z, F)$ of the graph $G = (X, F)$. We assume G to be a tree and as introduced above associate a game Γ in normal form with the n–person positional game on G. With G_z we associate a subgame as follows:*

The players priority sets in the subgame Γ_z are determined by the rule

$$Y_i^z := X_i \cap X_z, \ i = 1, 2, \ldots, n,$$

the set of the final positions

$$Y_{n+1}^z := X_{n+1} \cap X_z,$$

player P_i's payoff $H_i^z(x)$ in the subgame is taken to be

$$H_i^z(x) := H_i(x) \ , \ x \in Y_{n+1}^z \ , \ i = 1, 2, \ldots, n.$$

Accordingly, player P_i's strategy u_i^z in the subgame Γ_z is defined to be the truncation of player P_i's strategy u_i in the game Γ to the set Y_i^z, i.e.

$$u_i^z(x) := u_i(x) \ , \ x \in Y_i^z \ , \ i = 1, 2, \ldots, n \ .$$

Each subgraph G_z is associated with the subgame in normal form

$$\Gamma_z = \langle P_1, P_2, \ldots, P_n, U_1^z, U_2^z, \ldots, U_n^z, K_1^z, K_2^z, \ldots, K_n^z \rangle \ ,$$

where U_i^z *is the set of all strategies for player P_i in the subgame, and*
K_i^z *is the payoff–function of* $K_i^z : \prod_{i=1}^{n} U_i^z \to \mathbb{R}, \ i = 1, 2, \ldots, n \ .$

In chapter 2 we introduced the notion of a Nash–equilibrium in a noncooperative n–person game in normal form. Now we will strengthen this notion.

Definition 4.9. *The Nash–equilibrium $u^* = [u_1^*, \ldots, u_n^*]$ is called an* **absolute Nash–equilibrium** *in the game Γ if for any $z \in X$ the situation $(u^*)^z = [(u_1^*)^z, \ldots, (u_n^*)^z]$, where $(u_i^*)^z$ is the truncation of strategy u_i^* to the subgame Γ_z, is a Nash–equilibrium in Γ_z.*

Example: Lets consider the two–person game $\Gamma = \langle P_1, P_2, U_1, U_2, H_1, H_2 \rangle$ on the tree below. Consider the strategies

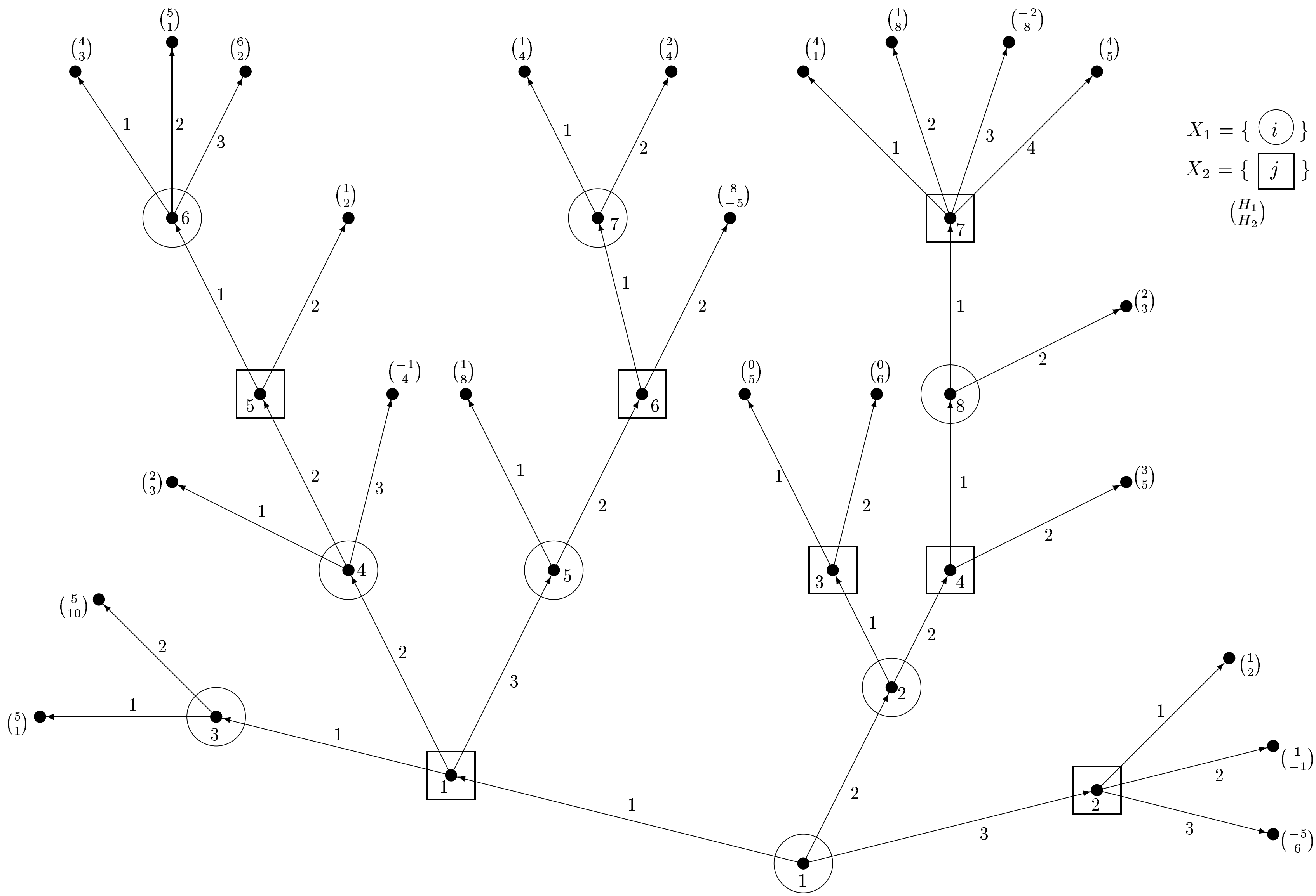
$$\begin{aligned} u^1 &= [2,1,2,3,1,2,1,1] \text{ for } P_1 \\ \text{and } u^2 &= [3,1,1,2,2,1,4] \text{ for } P_2 \end{aligned}$$

The situation $[u^1, u^2]$ leads to the final node with

$$\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \quad .$$

P_1 has $3 \cdot 2 \cdot 2 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 2 = 864$ strategies at all.
P_2 has $3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 4 = 576$ strategies.
In normal form Γ is a bi–matrix game with two payoff–matrices of 864×576–type each.

Theorem 4.1. *In any n-person positional game with complete information on a finite tree there exists an absolute Nash–equilibrium.*


$X_1 = \{ (i) \}$
$X_2 = \{ [j] \}$
$\binom{H_1}{H_2}$

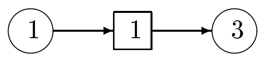
Remarks:

1. Theorem 4.1 ensures in the bi–matrix game connected with our example the existence of a Nash–equilibrium in pure strategies.

2. If Γ is a two–person zero–sum game (i.e. $H_1(x) + H_2(x) = 0, \quad \forall x \in X_{n+1} = X_3$) then in the corresponding matrix game the matrix has a saddle point.

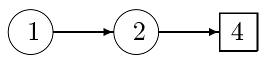
3. So, for instance in the matrix of chess a saddle point exists and therefore chess has a value (in the sence of the value of the saddle point) and both players have pure optimal strategies.

4. We can solve our example by the backward construction proposed in the proof of theorem 4.1:
   $[u^{1*}, u^{2*}]$ with $u^{1*} = (1,2,2,2,2,3,2,1)$, $u^{2*} = (1,3,2,2,1,1,2)$ is an absolute Nash–equilibrium. In the situation $[u^{1*}, u^{2*}]$ the game follows the path

   .

   In this equilibrium the payoffs to the players are different (5 resp. 10). But $[\bar{u}^{1*}, \bar{u}^{2*}]$ with $\bar{u}^{1*} = (2,2,1,1,2,3,2,1)$ and $\bar{u}^{2*} = (3,3,2,2,2,1,3)$ form an absolute equilibrium also. This situation follows the path

   .

   The payoffs now are again different (3 resp. 5) and less for both players than those in situation $[u^{1*}, u^{2*}]$.

Theorem 4.2 (Rochet 1980). *Consider an n–person positional game with complete information. Let the player's payoffs $H_i(x)$, $i = 1,2,\ldots,n$, in this game be such that*

if there exists an i_0 and there are x, y such that $H_{i_0}(x) = H_{i_0}(y)$ then $H_i(x) = H_i(y)$ for all $i \in \{1,2,\ldots,n\}$.

Then in the game the player's payoffs coincide in all absolute equilibria.